

DIAGRAMMATIC REASONING SKILLS OF PRE-SERVICE MATHEMATICS TEACHERS

Margaret Karrass

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ABSTRACT

Diagrammatic Reasoning Skills of Pre-service Mathematics Teachers

Margaret Karrass

This study attempted to explore a possible relationship between diagrammatic reasoning and geometric knowledge of pre-service mathematics teachers.

Diagrammatic reasoning skills, as a sequence of steps from visualization, to interpretation, to formalisms, are at the core of teachers' content knowledge for teaching. However, there is no course in the mathematics curriculum that systematically develops diagrammatic reasoning skills, except Geometry.

In the course of this study, a group of volunteers in the last semester of their teacher preparation program were presented with "visual proofs" of certain theorems from high school mathematics curriculum and asked to prove/explain these theorems by reasoning from the diagrams. The results of the interviews were analyzed with respect to the participants' attained van Hiele levels.

The study found that participants who attained higher van Hiele levels were more skilled at recognizing visual theorems and "proving" them. Moreover, the study found a correspondence between participants' diagrammatic reasoning skills and certain behaviors attributed to van Hiele levels. However, the van Hiele levels attained by the participants were consistently higher than their diagrammatic reasoning skills would indicate.

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CHAPTER 1 – INTRODUCTION

This study seeks to explore a possible relationship between geometric knowledge of pre-service secondary mathematics teachers and their visual reasoning skills. More specifically, when pre-service mathematics teachers are presented with diagrams of certain theorems found in high school mathematics curriculum, are they able to recognize these theorems from the diagrams? Can they reason from a diagram, thereby connecting visual and abstract representations? Is there a relationship between their ability to reason from a diagram and their knowledge of Geometry?

This study employs mixed methodology to explore and analyze how prospective teachers go about explaining/proving (Hanna, 2000) given theorems. Study participants are teacher candidates in the graduate Mathematics Education program at a major university, who volunteered to participate in this study.

This chapter includes an overview of the context that frames this study, the purpose of the study, and the research questions followed by a brief introduction of the research methodology.

Need for the Study

Visual reasoning skills are of great importance to mathematicians, scientists, engineers, architects, computer scientists, physicians, artists, and others, too numerous to name. Hoffman writes that visual-spatial intelligence is at the core of who we are. (Hoffman, 1998) Our visual reasoning skills are not just about our ability to see and “construct” visual images in our “mind’s eye”, but also our ability to see in the sense of understanding. Lakoff

and Núñez (2000) argue that our ability to perceive visually or through touch enables us to construct linguistic metaphors. Whiteley (2002) writes that according to research in intelligence testing, spatial temporal reasoning is positively correlated with superior performance in mathematics. Davis (1990) argues that “...mathematical discovery is not usually made in a deductive way...”, and that “...the ‘eye’ is the legitimate organ of discovery and inference.” Moreover, he emphasizes the need for mathematics educators “...to come to terms with those aspects of mathematics that are required by physicists, engineers, etc. and of the criteria by which these related professions validate their work ...”, that is, visualization and visual spatial reasoning. While studying brain processes with the use of fMRI¹ Anderson and colleagues (Terao, et al, 2004) asked adult subjects to solve verbal algebraic problems commonly found in high school curriculum. His conclusion was: “Mathematical thinking emerges from the interplay between symbolic and visuo-spatial systems. Algebra word problems, which are widely used in current school curriculum, are not a pure language processing task. They appear to depend on the use of visuo-spatial systems.” (p. 6)

“...The quality of mathematics teaching depends on teachers’ knowledge of the content ...”. (Ball et al., 2005) Numerous research studies have shown that “... the knowledge of the subject matter is an essential component of teacher knowledge”. (Ball, McDiarmid,

¹ Developed in the 1990’s, fMRI is a magnetic resonance imaging tool used to measure blood flow to the brain while it is performing cognitive tasks. The device works like a conventional MRI except that it uses two other phenomena, namely, when a given center in the brain is activated the blood vessels around it dilate resulting in an increased blood flow; and oxygenated blood carries more iron which distorts the magnetic field. “Functional MR Imaging (fMRI) – Brain”, RadiologyInfo.org, developed jointly by American College of Radiology and Radiology Society of North America. Accessed on 11-18-11 at <http://www.radiologyinfo.org/en/info.cfm?pg=fmribrain>.

1990) Moreover, “The mathematical knowledge needed for teaching is not less than that needed by other adults. In fact, knowledge for teaching must be detailed in ways unnecessary for everyday functioning. In short, a teacher needs to know more, and different, mathematics—not less.” (Ball, et. al., 2008, p. 396) Hill et al., write that the quality of teacher mathematical knowledge for teaching shows positive effect on student achievement. (Hill, et. al., 2005) Teachers’ comprehensive understanding of the subject increases students’ opportunities to learn. (Ma, 1999) Brown & Borko (1992), Usiskin (2002, April), Willingham (2003-2004, Winter) call for richer and deeper understanding of Mathematics among pre-service teachers. Clearly, in order to be able to develop visual reasoning skills in the new cadre of mathematicians, scientists, engineers, doctors, graphic designers, etc., Mathematics teachers should possess visual reasoning skills necessary for teaching. That is, visual reasoning skills, as a sequence of steps from visualization, to interpretation, to formalisms, are at the core of teachers’ content knowledge for teaching.

In his call for a research program to study visual-spatial reasoning, Whiteley (2004) posited a number of questions about students’ and their teachers’ visual thinking. Among them he asked whether teachers are able to do mathematics by means of visual reasoning. Perhaps, a sub-question implied by Professor Whiteley is about the factors in academic preparation that contribute to mathematics teachers’ visual reasoning skills. Could geometric knowledge be a factor in predicting mathematics teachers’ visual reasoning skills?

As important as visual reasoning may be, whether for developing professional knowledge or for study of Mathematics, there is no specific subject in the school curriculum that systematically develops in students this skill, except perhaps, for Geometry. Mathematical

folklore tells us that “to do Geometry is to reason accurately from an inaccurate drawing”. Indeed when solving a geometric problem we first examine or construct a diagram which describes the problem and then “argue from the picture”. We conjecture or perform thought experiments, which eventually, become translated into formal statements we call proofs. “This translation from visual to verbal suggests a possible method of moving from visual mathematics to formal mathematics. What is required is the ability to see the general in the particular images² to give meaning to the corresponding formal definition and to use the resulting links between imagery and formalism to formulate and prove theorems.” (Pinto, Tall, 2002)

For the purpose of this study we will define visual reasoning skills as those skills which allow us to proceed from an image to an understanding of the concept this image represents. Then visual reasoning is a process of refinement of schemas from a picture to a verbal statement.

This definition is rooted in the theory of Gray and Tall (1994) regarding the three worlds of Mathematics, namely, the embodied, the proceptual, and the formal worlds:

[The embodied world] – “... includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuospatial imagery”; [the proceptual world] – “...the world of symbols that we use for calculation and manipulation in arithmetic, algebra, calculus and so on”; [and the formal world] – “... based on properties, expressed in terms of formal definitions that are used as axioms to specify mathematical structures”. (Tall, 2004)

² Here Pinto and Tall quote Mason’s & Pimm’s 1984 paper “Generic examples: seeing the general in the particular” published in the *Educational Studies in Mathematics*, 15, 277–289.

Such definition of visual reasoning skills also reflects van Hiele's theory of levels of geometric thought. According to the van Hieles, geometric thought develops in sequence of five levels where each level requires re-organization or a refinement of the knowledge acquired at the previous level. The beginner's level is that of visualization, when a learner is able to recognize geometric shapes by naming them. This level is followed by analysis, then informal deduction, followed by deduction, and finally, by rigor, when a learner is able to conceive of various geometries. Although learner's progression through the levels is not necessarily linear, the object of perception at a lower level becomes the object of conception of the next higher level (van Hiele, 1986) as echoed by Gray and Tall.

It follows then that an individual, functioning at a higher level of geometric thought as described by van Hiele, should possess stronger visual reasoning skills as defined above; and since progression through van Hiele levels is achieved through instruction in Geometry visual reasoning may be improved by geometric instruction as well.

Although recently there has been a lot of interest in elementary in-service and pre-service teachers' knowledge of Geometry (Hill, Rowan, Ball, 2004 & 2005; Goulding, Rowland, & Barber, 2002; Stacey, Steinle, & Irwin, 2001; Galuzzo, Leali, & Loomis, 2000; Ma, 1999; Battista, et.al., 1982) just to name a few, there has been no similar attention paid to the nature of geometric knowledge of secondary in-service or pre-service teachers. (Chinnappan, Lawson, 2005)

In 2003 the Organization for Economic Cooperation and Development (OECD) released the results of its Program for International Student Assessment (PISA) among the 15-year-

olds. (PISA, 2003) The United States performed below the OECD average on each mathematics literacy subscale representing a specific content area, among them, space and shape. (Lemke et al. 2001)

Such results are not surprising possibly due to several factors. One of them is that geometry component of the mathematics curriculum has consistently been crowded out by other areas of mathematical studies, i.e., greater emphasis on algebra, inclusion of probability and statistics, etc. Another factor may be attributed to the growth of various geometries, and in view of such growth, the inability of curriculum designers to create a comprehensive geometry curriculum. Finally, to paraphrase Felix Klein's expression of "double-forgetting"³ as "double-not-knowing", with my apology to Professor Klein, the fact that mathematics teachers returning to high schools to teach mathematics know negligibly more geometry than the students they are about to teach. (Jones, 2000)

The NCATE/NCTM⁴ Program Standards (2003), more specifically, Standard 11: Knowledge of Geometries calls for "...[teacher] Candidates [to] use spatial visualization and geometric modeling to explore and analyze geometric shapes, structures, and their properties." (p.5) This standard consists of eight performance indicators which require

³ As quoted by Hans Freudenthal "double-forgetting" refers to prospective teachers first forgetting high school mathematics as they enter university and then forgetting what they have learned in university upon returning to teach high school mathematics (Freudenthal, 1973).

⁴ NCATE - National Council for Accreditation of Teacher Education is a coalition of member organizations. It was established in 1954 as a professional accrediting body for teacher preparation.

NCTM – National Council of Teachers of Mathematics is a professional association of teachers. It provides guidance and resources for developing and implementing mathematics curriculum, instruction, and assessment. The organization promotes equity in mathematics education, engages in public and political advocacy, provides professional development, and encourages the integration of research and practice in mathematics education.

teachers to exhibit knowledge in formal structures, properties of geometric objects (symmetry, congruence, and similarity), transformations, applications, connections, Euclidean and non-Euclidean geometries, historical development, and philosophical implications. Moreover, this standard calls upon teacher candidates to “Use concrete models, drawings, and dynamic geometric software to explore geometric ideas and their applications in real-world contexts”. (ibid) However, prospective teachers entering teacher education programs often have taken only one upper-level course in non-Euclidean Geometry during their undergraduate studies; and many teacher-candidates may not have experienced Geometry instruction during undergraduate studies at all. After all, NCATE/NCTM Professional Standards for Teaching Mathematics require of prospective Mathematics teachers merely an equivalent of a major in Mathematics, just enough “...to gain sufficient understanding of the Mathematics standards for teaching”. (ibid., p. 139) Let us consider possible student streams that contribute to the Mathematics Education programs. Generally, these are of three types:

- a. Mathematics majors who enter an undergraduate Mathematics program intending to become Mathematics teachers;
- b. Mathematics majors who enter an undergraduate program and focus on particular areas of inquiry in Mathematics, for example, Operations Research or Analysis, intending perhaps, to become actuaries; for whom a Mathematics Education program is somewhat of an afterthought; and finally,
- c. Students who major in engineering, computer science, business, finance, etc., and take enough courses in Mathematics to qualify for admission to a Mathematics teacher preparation program.

Only teacher candidates from the first stream (a) are likely to take a sequence of undergraduate courses that emphasizes the breadth of experience in the study of Mathematics. Since an undergraduate Geometry course will not be part of their degree

requirements, teacher candidates from the other two streams (b) and (c) will possess geometric knowledge on the level of high school at best. The deficiency of such geometric knowledge was emphasized by Wirszup (1976) who wrote that a high school geometry course requires a higher level of thought than most high school students generally possess and that “this irreparable deficiency haunts them continually later on”.

However, one might argue that prospective teachers might gain geometric knowledge while taking a course in Calculus, Differential Equations, or perhaps, a course in Linear Algebra, which for decades has been a core requirement for mathematics majors and is often a recommended course in sciences, engineering, and business programs.

Hans Freudenthal argued that Linear Algebra is not well suited for developing “locally axiomatic” system, such as, for example, Euclidean Geometry, because Linear Algebra is a well-structured, more rigorous, a priori organized, “globally axiomatic” system, which requires no organizing on the part of the learner. (Freudenthal, 1973) In his comments to the Committee on the Undergraduate Programs in Mathematics, Professor Wu echoed Freudenthal by suggesting that prospective Mathematics teachers should become fluent with “proofs by local axiomatics”. (CUPM, 2004) Banchoff and Wermer (1992) write that the study of Geometry should precede the study of Linear Algebra: “Our experience in teaching undergraduates over the years has convinced us that students learn the new ideas of linear algebra best when these ideas are grounded in the familiar geometry of two and three dimensions. ... Moreover, we feel that this geometric approach provides a solid basis for the linear algebra needed in engineering, biology, and chemistry, as well as in economics and statistics”. (p. ii)

Since visual reasoning skills are of great importance to mathematicians and non-mathematicians alike; since visual reasoning skills are part of teachers' content knowledge for teaching; since these skills may be systematically developed through the study of Geometry; and since prospective secondary school Mathematics teachers may not have sufficient preparation in Geometry it behooves us to explore a possibility of a relationship between teachers' ability to reason from diagrams and their knowledge of geometry. More specifically, this study will attempt to determine if a relationship exists between pre-service teachers' knowledge of Geometry and their ability to recognize a "concept image" of a theorem – a "visual proof" and transform an embodied representation of a mathematical concept into a formal statement about this mathematical concept. (Tall, 2004)

This study will attempt to answer the following research questions:

1. Is there a relationship between visual-spatial skills, van Hiele levels of geometric thought, and the academic experience in Geometry of pre-service secondary mathematics teachers?
2. Is there a relationship between van Hiele levels and diagrammatic reasoning skills of pre-service high school mathematics teachers as they prove/explain visual theorems?
3. Does the van Hiele model adequately describe pre-service high school mathematics teachers' knowledge of geometry?
4. Does the van Hiele model adequately describe pre-service high school mathematics teachers' knowledge of geometry for teaching?

Procedure of the Study

To answer research questions posited in the preceding section a mixed design methodology was used. More specifically, a Triangulation Design – Convergence Model approach (Creswell, Plano Clark, 2007) was employed to collect and analyze data.

Data collection occurred in two consecutive stages. During the first stage, data pertaining to academic experience, visual-spatial skills, and van Hiele levels of geometric thought was collected through a series of questionnaires and tests. During the second stage, which followed immediately, the participants, were shown diagrams of selected theorems and asked to “argue from the diagrams” during a semi-structured interview. The interviews were audio-taped.

Data analysis occurred in three stages. Collected quantitative data were analyzed separately. Qualitative data were transcribed, coded, and analyzed. Both types of data were compared and interpreted.

Chapter 2 contains the review of literature pertinent to this study. Chapter 3 contains detailed description of the study methodology, and procedures. Chapter 4 of this manuscript contains findings and analysis of the results. Discussion of findings may be found in Chapter 5; and Chapter 6 contains summary, conclusions, and suggestions for further research.

CHAPTER 2 – LITERATURE REVIEW

This chapter examines research literature relevant to the investigation of a relationship between diagrammatic reasoning skills and geometric knowledge of pre-service secondary mathematics teachers. To answer the research questions several areas of scholarship germane to the issues under the investigation will be discussed, namely, historical and philosophical perspectives pertaining to diagrammatic reasoning in mathematics, research on van Hiele levels of geometric thought, current thought on some cognitive aspects of mathematical knowledge, and teacher knowledge for teaching. In addition, definition of diagrammatic reasoning will be developed and several research studies of diagrammatic reasoning will be discussed with the purpose of establishing theoretical framework for data analysis.

Historical-Philosophical Perspective

Contemporary mathematics instruction is deeply rooted in constructivist theory. Due to the work of Kant, Dewey, Piaget, Vygotsky, Bruner, Morin, and many others, we have come to believe that humans construct knowledge by internalizing their experiences. Consequently, mathematics teachers are expected to function as facilitators of learning experiences or guides on students' paths of discovery, employing a variety of techniques in helping students "discover" mathematics and "construct" meaning of mathematical objects. Constructivist approach emphasizes the learning process, i.e., exchange of ideas, problem solving, "... a [classroom] culture, in which students are involved not only in discovery but in a social discourse involving explanation, negotiation, sharing, and evaluation." (Kamii & Lewis, 1990)

Research indicates that students develop better understanding of mathematics when teachers take on a constructivist approach to instruction rather than a traditional approach. (Cobb, Wood, Yackel, & Perlwitz, 1992) In adopting constructivist practices a teacher uses a variety of tools for mathematical discovery, concept building, and communication. However, certain types of “discovery”, for example, image-based or diagram-based reasoning were not always *en vogue*. The following section briefly examines the evolution of attitudes towards diagrammatic reasoning among mathematicians.

According to Hoffmann, Charles Saunders Peirce⁵, a logicist and a philosopher, believed that reasoning with the aid of diagrams is essential to mathematics and that there is no mathematical reasoning that is not diagrammatic. He quotes Peirce: “Mathematical reasoning consists in constructing a diagram according to a general precept in observing certain relations between parts of that diagram not explicitly required by the precept, showing that these relations will hold for all such diagrams, and in formulating this conclusion in general terms. All valid necessary reasoning is in fact thus diagrammatic.” (Peirce CP 1.54/Hoffmann, 2003) Diagram-based reasoning or *diagrammatic reasoning* may be defined as that, which is facilitated or mediated by a *visual representation*, a graphically rendered cognitive construct, i.e., “...formal system for making explicit certain entities or types of information ...” (Marr, 1982), a gestalt view.

When “doing mathematics”, a mathematician may casually sketch a diagram to support her reasoning; however, the diagram gets quickly discarded for the fear that it may lead one’s

⁵ Charles Sanders Peirce (1839–1914), a polymath and a highly prolific writer was the founder of American pragmatism. Peirce referred to his philosophy as “pragmatism” to emphasize the difference between his views and those of others also labeled “pragmatism”. (Burch, 2010)

reasoning astray. A mathematician is more willing to introduce a diagram when dealing with “spatial” or “geometric” ideas, although Greaves (2002) argues that historically the role of diagrams in geometric reasoning depends on what is contemporaneously considered to be the subject-matter of Geometry. However, intentionally omitting diagrams from mathematical research writing is a fairly modern idea.

The ancient Greeks believed diagrams to be essential in geometric proofs. Notwithstanding the use of magnitudes, the Greeks did not employ, in the same manner the Egyptians did, fixed units for measurement in finding geometric relationships (Netz, 1999). For assigning measures to geometric entities would reduce the study of Geometry to specific cases, and thus Geometry would cease being a deductive study concerned with general results (Mueller, 1981). Sir Thomas L. Heath in the Introduction to his 1908 edition of Euclid’s *Elements* quotes Aristotle from *Posteriori Analytics*: “... ‘...the geometer falsely calls a line which he had drawn a foot long when it is not, or straight when it is not straight.’ The geometer bases no conclusion on the particular line which he has drawn being that which he has described, but (he refers to) what is illustrated by the figures.” (Heath, 2006)

For centuries mathematicians studied and discovered mathematics aided by images and diagrams. In his *Regulae ad Directionem Ingenii*⁶, René Descartes (1684/1997) wrote: “[Rule 12]...nothing falls more readily under sense than figure, which can be touched and seen...” (p.40) and that “[Rule14] The same rule is to be applied also to the real extension of the body. It must be set before the imagination by means of mere figures, for this is the best way to make it clear to the understanding.” (ibid., p. 59)

⁶ *Rules for the Direction of the Mind* was published posthumously in 1684, in Dutch.

E.T. Bell (2008) writes that “The creators of calculus, [Newton and Leibniz], including Fermat, relied on geometric and physical (mostly kinematical and dynamical) intuition to get them ahead: they looked at what passed in their imaginations for the graph of a ‘continuous curve’, pictured the process of drawing a straight line tangent to the curve ...” (p. 59)

Blaise Pascal used diagrams in explaining and justifying some properties of the arithmetic triangle he discovered when working on mathematics of probability. Carl Boyer writes that Pascal’s explanation of newly discovered properties of the arithmetic triangle⁷ is significant not only because of the properties themselves, but because it was “...an eminently clear-cut explanation of the method of mathematical induction⁸ : “In every arithmetic triangle, if two cells are contiguous in the same base, the upper is to the lower as the number of cells from the upper to the top of the base is to the number of those from the lower to the bottom inclusive.” (Boyer/Merzbach, 1989)

Boyer further writes that Pascal called positions in the vertical column “cells of the same perpendicular rank” and those in the horizontal rows “cells of the same parallel rank”. The “cells of the same base” were those positioned on the diagonal with the increasing slope as in (Figure 1). It is worth noting that without a diagram accompanying the description, Pascal’s reasoning is hard to follow, yet with a diagram the elegance and the brilliance of the result is immediately intuitively grasped.

⁷ The arithmetic triangle, known to us as Pascal’s triangle, was described by Girolamo Cardano, but this is not the first time we learn about it. The arithmetic triangle was known to Arabic mathematicians as early as 11th century B.C.E.

⁸ Boyer writes that the term “mathematical induction” was originated by De Morgan, Pascal, together with Fermat, developed “reasoning by recurrence” (Boyer, 1968, 1989)

1	1	1	1	1	1	1	
1	2	3	4	5	6		
1	3	6	10	15			
1	4	10	20				
1	5	15					
1	6						
1							

Figure 1 – Pascal’s Triangle

Sir Isaac Newton explains in *De Methodus Fluxionum et Serierum Infinitorum*, his method for finding solutions for $P(x, y) = 0$, where $P(x, y) = \sum_{ij} a_{ij} x^i y^j$ is a polynomial in x and y .⁹ He solves this problem by expressing y in terms of a series in x , i.e.,

$$y = \sum_{k \geq 0} b_k x^k.$$

“However, to make this rule more evident, I thought it fitting to expound it in addition with the aid of the following diagram...” (Whiteside, 1969)

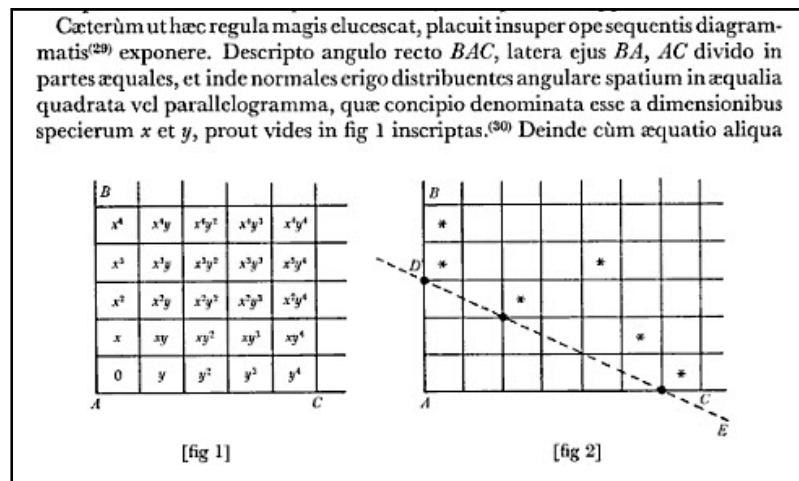


Figure 2 – Newton’s Parallellogram

⁹ This method later became known as “Newton’s Parallellogram” and when rediscovered by Victor Puiseux ca.1850 while working on a generalization of formal power series became known as Newton’s polygon.

Diagrams, attributed to John Venn (c. 1880), widely used in scientific writing, mathematics, statistics, linguistics, computer science, etc., may be traced back to Euler, to Leibniz before him, and as far back as Ramon Llull's *Ars Magna*. (Dunham, 1994)

Trying to make the notion of betweenness more precise, Moritz Pasch¹⁰ proposed the following axiom: “Let A, B, C be three non-collinear points, and let l be a line not containing any of A, B, C . If l contains point D lying between A and B , then it must also contain either a point lying between A and C or a point lying between B and C , but not both.” (Hartshorne, 2000, p. 74) In simpler terms, the axiom means that if a line intersects one side of a triangle and misses the three vertices, then it must intersect one of the other two sides. (Figure 3) This axiom becomes immediately clear and indisputable if one sees its graphical representation. (Weisstein, 2005b)

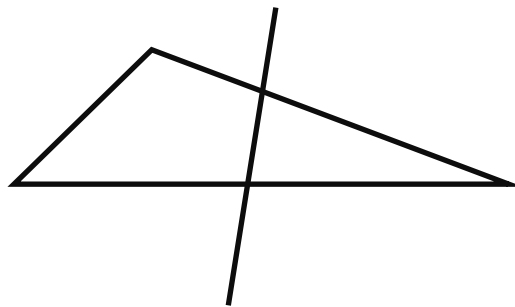


Figure 3 – Pasch's Axiom

¹⁰ Moritz Pasch (1843-1930), concerned with the tacit assumptions in Euclid, called for the formalization of Euclidean geometry. In his *Vorlesungen über neuere Geometrie*, published in 1882, Pasch argued that mathematical reasoning should not be based on physical or intuitive perceptions of space, but rather rely solely on formal manipulations. Pasch preceded and inspired the work of many mathematicians, among them Peano and Hilbert.

Hermann Weyl (2009) writes that next to Euclid's *Elements*, David Hilbert's *Grundlagen der Geometrie*, published in 1899, has become the second most influential and most read work on Geometry. He attributes Hilbert's success to several factors, among them, the use of diagrams: "... His [Hilbert's] axioms are stated clearly with a minimum of unfamiliar and unnecessary symbolism and many of his proofs are given in the Euclidean tradition with accompanying suggestive diagrams."

However, in the years that followed, mathematicians would come to reject diagrammatic reasoning changing mathematics education in the process. Two major factors contributed to the paradigm shift, namely, the formalization of mathematics and the Bourbaki.

Euclid's *Elements*, one of the finest intellectual achievements of the Ancient Greeks, gave Mathematics a method of organizing and deriving propositions, which have the highest degree of certainty accessible to human reason. For centuries the *Elements* "was a paragon of mathematical rigor". (Aberdein, 2006) However, upon examination, it was realized that Euclid's work, admirable as it was, contained logical gaps. There are many places, (beginning with *Elements I.1*), where Euclid's stated axioms and postulates are not sufficient to make his conclusions follow by formal logic alone. The most common criticism of Euclid is that his work is highly persuasive because a convincing diagram, which comfortably fills in the gaps in his reasoning, usually accompanies it.

Euclid's "parallel postulate" a non-obvious and unproved assertion, which Euclid himself was possibly ambivalent about since he waited until *Elements I.29* to first introduce it, challenged the Greeks, the Arabs, the Persians, and the Europeans alike for two thousand

years. From the 1820's, starting with the work of Gauss, Bolyai, Lobachevsky, and Riemann ending with the publication in 1899 of David Hilbert's *Grundlagen der Geometrie*, the first treatment of Geometry (in fact the first treatment of any branch of mathematics) presented in modern axiomatic form, mathematics evolved. Mathematicians understood that Euclidean geometry neither accurately reflected the structure of space nor was it a logical necessity, an innate, *a priori* intuitive part of our psyche as Kant thought it to be. (Penrose, 1989) Hilbert rejected the idea of "physically meaningful axioms". His concept of axioms was that they are purely formal relations among undefined elements. "Before Hilbert, nobody had yet brought out with as much determination and clarity the fundamental principle that in mathematics the precise nature of the entities studied does not matter; it is the relations between these entities, which alone are of importance." (Moore, 1985)

In Hilbert's vision Geometry first, and eventually, all of Mathematics would become nothing more than a manipulation of symbols, hence formalism. There were those who opposed Hilbert. Gottlob Frege, a logicist, insisted, in a letter to Hilbert in December 1899, on the traditional view of geometric axioms as following from the spatial intuition. (ibid.) Notwithstanding, Hilbert and his followers continued developing Foundations of Mathematics. Jeanne Dieudonné (1989) wrote that the axiomatic method, of which Hilbert was the champion, has revealed unsuspected analogies and permitted extended generalizations; and that the origin of the modern development of algebra, topology and group theory is to be found only in the employment of the axiomatic method. Mathematics transformed from the study of problems into a study of models resulting in an explosion of new fields within Mathematics. Famous and highly influential group of predominantly

French mathematicians, who functioned and wrote under the name of Nicolas Bourbaki, stated that: "... the internal evolution of mathematics has, in spite of appearances, tightened the unity of the various parts more than ever, and has created sort of central kernel, more coherent than ever. The essence of this evolution has consisted in systematization of the relations existing among the various mathematical theories, and is comprised in an approach generally known under the name of 'axiomatic method'."(Bourbaki, 1989)

The tidal changes within mathematics eventually began to influence mathematics education. Mashaal (2006) writes that at the turn of the twentieth century David Hilbert and Henri Poincaré dominated the world of mathematics, representing the German and the French schools of mathematics respectively. Their philosophical approaches to mathematics could not have been more different.

According to E. T. Bell (2009), Henri Poincaré, perhaps "the last great mathematical universalist", was endowed with great geometric intuition and exceptional visual-spatial skills, which in part informed his philosophical view of mathematics. Unfortunately, he did not leave direct (mathematical) descendants. (p. 527) Moreover, according to Dieudonné (Mashaal, p. 16) World War I claimed a great number of French mathematicians who could have followed in Poincaré's footsteps had they survived. In contrast, Germany was able to save most of its cultural heritage and Hilbert's school became highly influential. Consequently, the young cadre of Frenchmen coming up in the late 1920's and 1930's left France to study in England, or the United States, but mostly in Germany. Bourbaki admitted that although they recognized the genius of Poincaré they disliked his style and were much more influenced by Hilbert. (ibid., p. 47)

Mashaal writes that Bourbaki saved French mathematics from extinction. (ibid., p. 45) Initially, Bourbaki was formed to write a suitable text in Calculus, but the project evolved into an exposition of Mathematics, of the magnitude of Euclid's Elements. Conceptually, Bourbaki's *Éléments De Mathématique*, (mathematics in the singular, emphasizing unity), rested on three key ideas, namely, "... unity of mathematics, axiomatic method, and the study of structures". (ibid., p. 71) Regrettably, a comprehensive discussion of Bourbaki contribution to mathematics is beyond the scope of this review; here my interest in Bourbaki relates to the group's influence on mathematics education.

Starting with famous "Euclid must go!" challenge by Jean Dieudonné¹¹ in 1959 and for the next ten years Bourbaki transformed mathematics education in France with many other countries around the world to follow, including the United States. Sixty-five years later and a compendium of texts, in practically every area of mathematics, even in history of mathematics, written in a very particular style, influenced the way mathematics was taught from elementary to graduate school. The group's philosophy of mathematics was reflected in texts they created, i.e., from general to specific, limited number of diagrams, limited number of examples, great number of definitions, new terminology, new notation, etc.; these texts informed mathematics curriculum. Mashaal writes that it was not the most brilliant idea from the pedagogical point of view because Bourbaki mathematics could be read and understood either by seasoned mathematicians or with an aid of copious notes written to accompany standard texts. He quotes Bourbaki: "... the reader may not

¹¹ Jean Alexandre Eugène Dieudonné (1906 – 1992) was one of the founders and active members of the Bourbaki group. During a conference dedicated to reforming content and methodology of secondary mathematics education he exclaimed "Euclid must go!" causing many members to walk out.

understand the use of some of the material until later chapters unless he already has a reasonably broad knowledge of mathematics.” (ibid., p. 55)

In general terms curricular changes inspired by the Bourbaki, included the principles of mathematical and formal logic, naïve set theory, introduction to groups, rings, fields, and vector spaces presented axiomatically; linear algebra replaced geometry, and so on. The significance of the reform was in the increased rigor of the mathematics taught. Mathematics was to be presented as a well-organized, globally axiomatic system emphasizing proof-writing and de-emphasizing computational and other intuitive tasks. (ibid, p. 141) Figure 4 illustrates a definition of the angle between two rays, as taught in 1971 to eleven-graders. It was not accompanied by a diagram. (ibid.)

THEOREM AND DEFINITION

For any two pairs (D_1, D_1') and (D_2, D_2') of vector rays in E_2 the relation “There exists a vector rotation f of E_2 such that $f(D_1) = D_1'$ and $f(D_2) = D_2'$ ” is an equivalence relation in $D \times D$ where D is the set of vector rays in E_2 . An equivalence class for this relation is called **an angle of two vector rays** in E_2

Figure 4 – Definition from a Mathematics text for 11-graders

The new approach to teaching mathematics had its opponents on both sides of the Atlantic¹². Various philosophical and pedagogical objections have been raised. Philosophically, the difference between the “old” and the “new” approaches to

¹² Vladimir Igorevich Arnold (1937 – 2010), when writing about Isaac Barrow, delivered a stinging criticism of Bourbaki: “What did Barrow's lectures contain? Bourbaki writes with some scorn that in his book in a hundred pages of text there are about 180 drawings. (Concerning Bourbaki's books it can be said that in a thousand pages there is not one drawing, and it is not at all clear which is worse.)” (Arnold, 1990)

mathematics was that between Platonism and Formalism. Reuben Hersh (1999) writes that although it is difficult to document a connection between formalism in expository style of mathematical research and formalism in philosophical attitude, ideas have consequences. “What I think mathematics is affects the way I present it.” (p. 41)

“ ... [A] man is an ideological animal by nature ...” , following Althusser’s (1970) reasoning, mathematicians, just as other humans [*sic.*], are *always-already interpellated*¹³, that is, they are “born as mathematicians” into an ideology pre-determined by their education, their perception of the world, and their beliefs about mathematics, which are acquired through personal experience in mathematical inquiry and practice. To remove any political connotation we can substitute the word ideology with a more neutral – philosophy of mathematics. Modern philosophical views are highly nuanced, but for the purpose of this study it is sufficient to accept a greatly simplified view of the currents in Philosophy of Mathematics and identify two main streams, namely, *realism* and *formalism*. More specifically, we will define, with very broad strokes, realism as an umbrella term for intuitionism and Platonism, but recognize that there are other philosophical theories that fall under this umbrella, such as empiricism, logicism, etc.

Hersh writes that both Platonism and Formalism, on opposite ends of the spectrum, ignore the question of teachability [*sic*] of mathematics, and therefore, are not adequate philosophies. “If mathematical objects were an other-worldly, non-human reality

¹³ Interpellation is a process whereby individuals recognize themselves being subjugated by an ideology.

(Platonism), or symbols and formulas whose meaning is irrelevant (formalism), it would be a mystery how we can teach it or learn it.” (Hersh, p. 238)

Among the harshest critics of the formalist movement in mathematics education was Hans Freudenthal, especially, regarding the teaching of geometry. In his *Mathematics as an Educational Task*, Freudenthal (1973) makes a strong case for a more intuitive approach. He writes: “Geometry ... was always considered more as a discipline of mind than any other part of mathematics for it could boast of closer relations to logic. ... [However], if geometry as a logical system is to be imposed upon a student it would indeed be better abolished. ... Geometry can only be meaningful if it exploits the relation of geometry to the experienced space. ... Geometry is one of the best opportunities that exist to learn how to mathematize [*sic*] reality.” (p. 406)

Presently, some fifty years later, researchers are becoming increasingly interested in diagrammatic reasoning mostly due to the use of computers in mathematical research and in mathematics education. (Healy & Hoyles, 2001; Jones, 2000; Laborde, 2000, 2002; Mariotti, 2000) The NCTM recommends the use of computers in the study of mathematics in general and geometry in particular in Pre-K through Grade 12. (NCTM, 2000)

As a consequence of the Calculus Reform, many universities require a mandatory computer lab component in the Calculus sequence. When studying mathematics students are encouraged to experiment, observe, and interpret behavior of functions, or “discover” geometric properties of objects when using dynamic geometry environments, etc. Dynamic geometric environments allow students, through experiment and conjecture, discover

properties of objects in Euclidean Geometry. Such experimentation leads to the development of linguistic symbolization and verbalization (Presmeg, 1992), which in turn leads to the development of deductive reasoning and the development of heuristic strategies for problem solving (Schoenfeld, 1987). However, there is still a fair degree of scarcity with respect to peer-reviewed publications in diagrammatic reasoning, especially that of pre-service or in-service high school teachers. (Chinnappan, Lawson, 2005) This area of research, generally, is reduced to studies in visual-spatial abilities or skills.

Visualization and Diagrammatic Reasoning

In this section the reader will find a review of research literature that describes visualization and diagrammatic reasoning, with the purpose of establishing a working definition.

VISUALIZATION

Visual spatial skills are generally considered to be of two types, namely visualization (Carroll, 1993) and spatial orientation (Tartre, 1990). The name “visualization” implies seeing in the “mind’s eye” and mentally manipulating objects through translations, rotations, reflections, etc. Once considered a purely physical act requiring innate abilities, visualization was shown to involve the use of analytic strategies. (Geisser, Lehmann, & Eid, 2006; Hegarty & Waller, 2006) These strategies include task decomposition and rules-based reasoning. (Hegarty, 2010) Presmeg (1986) and Aspinwall (1997) describe visualization as a process involving perception, manipulation and analysis of visual images.

Tartre (1990) identified *spatial orientation* skills of particular importance for problem solving. In her study of spatial orientation skills and mathematical problem solving of

tenth-grade high school students she argues that spatial orientation tasks could involve organizing, recognizing, making sense out of a visual representation, re-seeing it or seeing it from a different angle, without mentally moving the object. Smith (1964, in Tartre, 1990) thought that the process of perceiving and assimilating a gestalt is about abstraction. He wrote: "... it is possible that any process of abstraction may involve in some degree the perception, retention in memory, recognition and perhaps reproduction of a pattern or structure". (Tartre, p. 213)

Tartre also quotes Franco and Sperry (1977) identifying two types of logical thinking processes, namely, a step-by-step rational, deductive, and often verbal process, and another structural, global, intuitive, spatial, inductive process. They found that non-verbal visual-spatial apprehension precedes and supports sequential deductive analysis involved in solving geometry problems. (Tartre, p. 112) Lorenz (1968) likened the gestalt perception to intuition when a concept "jumps out" from the background of irrelevant information. (p. 1976) Tartre equates this phenomenon with gaining insight. (ibid.)

Visual images may be perceived pictorially or schematically. Success in problem solving is positively correlated with schematic perception of images, while the lack of success is positively correlated with pictorial perception. (Tartre, 1990; Monk, 1992; Presmeg, 1992; Hegarty & Kozhevnikov, 1999) Mayer & Massa (2003) showed that proficiency in creating, holding, and manipulating spatial representations indicates high spatial ability while difficulties performing these tasks indicate low spatial ability.

DIAGRAMMATIC REASONING: DEFINITION

If visualization is about perceiving, manipulating, and analyzing images (Presmeg, 1986; Aspinwall, 1997) then visualization is a component of *visual* or *diagrammatic reasoning*. Hoffman (2007) makes a “terminological distinction” between visualization and diagrammatic reasoning; he calls the former mental modeling. (p. 7) Fischbein (1987) writes that visual reasoning “... not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.” (p. 101). Arcavi argues that an interpretation of visual clues may be prompted, guided, or supported by a symbolic representation given observer’s “... good dose of symbol sense”. Furthermore, he believes that visual processing involves some verbalization: “...visualization as a process is not intended to exclude verbalization ... quite the contrary, it may well complement it.” (Arcavi, 1994)

Hoffmann (2007) defines diagrammatic reasoning as a “thinking facilitation” process. He writes that having a diagram before one’s eyes helps, among other things, analyze a problem, clarify implicit assumptions, structure problem space, initiate ‘negotiation of meaning’ regarding the elements used in the diagram, motivate argumentation, etc. Moreover, it is about decision making and knowledge development (p. 6)

Arcavi (2003) defined visualization as a complex act, that is “ ... the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding.” (p. 217) Here we will adopt Professor Arcavi’s definition.

Definition: To engage in diagrammatic reasoning means to create or observe a graphically rendered cognitive construct; to perceive its components and inherent structures; to reflect upon these perceptions; to intuitively generate new hypotheses and verify them; to communicate ideas; and finally, to make connections.

DIAGRAMMATIC REASONING: COGNITIVE TOOL

Diagrammatic reasoning acts upon diagrams. Hoffmann (2007) defines it as: "... an external representation of relations that is constructed according to the rules and conventions of, and by means of the elements and relations available in, a certain system of representation. Such a representational system provides the means, and constrains the possibilities, both of constructing diagrams, and of any manipulation we perform on those diagrams." (p. 9)

Hoffman (1998) quotes Peirce "... diagrams reduce complexity thereby enabling concentration on essential relations, and revealing those elements that are the most promising candidates for context dependent interpretations". (p. 134) To construct a diagram, or to interpret a diagram, one must follow, according to Peirce, an abstract "precept". "Representing something in diagrams is possible only by using a certain language, and notation, or more generally, a certain representational system which has a rationality of its own, a certain syntax and semantic." (ibid.)

A *precept* is a mental representation of a perceived stimulus. A geometric diagram may be thought of as a *precept* or a collection of *precepts*. For example, a triangle may be thought of as a three-sided geometric shape; or as a collection of geometric shapes such as line

segments, angles, or vertices in some relationship to each other, i.e., a set of three non-collinear points. Each of these would be considered a *precept*. Each precept is then a Peircian triad consisting of a *sign* = image, an *object* = triangle, and an *interpretant* = three non-collinear points.

A construct consisting of precepts and of an accompanying linguistic representation of these precepts is defined as a *concept*. Since there may be a variety of linguistic descriptions associated with a given precept, just as there are many ways to describe an object or an event in common human communication (Anderson, 1978) – a precept to concept is a one-to-many relation.

Paivio's (2006) conception of reasoning, generally, involves two equally important and inter-dependent systems, namely, verbal – responsible for language processing and non-verbal, imagistic – responsible for non-linguistic processing. His research showed that the sub-systems can function either independently or cooperatively to mediate nonverbal and verbal behavior; that representational activity may or may not be experienced consciously as imagery and inner speech; and that the verbal system dominates in some tasks such as crossword puzzles and the nonverbal, imagistic system in others such as jigsaw puzzles. Paivio found that: "Cognition is this variable pattern of the interplay of the two systems according to the degree to which they have developed ..." (p. 3)

Diagrammatic reasoning is reported in the literature as a powerful cognitive tool. Kindfield's (1999) study showed that successful use of diagrams as tools "to think with" when reasoning, indicated participants' advanced conceptual knowledge. Specifically,

more advanced study participants systematically and consistently displayed a variety of diagram-related reasoning behaviors such as knowledge-dependent representational variability and fine-tuning of diagrams to the immediate reasoning task. These behaviors were limited or absent among the less advanced participants. (p. 82)

DIAGRAMMATIC REASONING: BENEFITS

In an experiment conducted by Ainsworth & Loizou (2003) subjects were given information about a circulatory system in a form of a diagram or printed text and prompted to “self-explain”. The results showed that students given diagrams generated more of self-explanation, and performed better on the post-test than students given textual information. The authors of the study concluded that diagrammatic reasoning is an effective meta-cognitive tool for self-explanation and that it can help students develop deeper understanding of the material.

Larkin and Simon (1987) distinguished the role of diagrams in three separate processes: search, recognition, and inference. They found that when subjects scan an image and examine its elements they are able to find information much faster than if the same information is presented verbally. They also showed that diagrams make it easier to identify instances of a concept. That is, an iconic representation is recognized faster than a verbal description reducing the number of cases in need of examination. However, the affect of diagrams on the process of inference is not as strong as on search and recognition. Larkin and Simon conclude that inference is largely independent of representation. (p. 71)

Hoffmann (2007) argues that the function of diagrammatic reasoning may be best described by a metaphor of “scaffolding” and as such fits well within the “distributed cognition” framework. (p.18) Zhang & Patel (2006) describe distributed cognition as a discipline concerned with processes distributed across internal minds (i.e., representations, perceptions, memory, meta-thinking, etc.) in relation to external artifacts, such as diagrams. Artifacts possess properties that enhance internal mind. They may function as memory enhancers, provide information unavailable from internal representations, support perception, facilitate recognition, anchor and structure cognitive behavior, support perceptual rehearsal making invisible and transient information visible and sustainable, aid processing by limiting abstraction, facilitate abstraction by going from concrete to general, etc. (p. 6)

DIAGRAMMATIC REASONING: DIFFICULTIES

Arcavi identifies two possible cognitive difficulties related to visual reasoning, namely, high cognitive demand, and flexibility of translation between visual and analytic representations. Regarding the former, interpreting diagrams may be modality dependent, i.e., stronger visual-spatial ability may be a factor of success. Visual inspection may be hindered by irrelevant information. Presmeg (1986) reports that when viewing a diagram subjects may pay attention to irrelevant details. (p. 44) Larkin and Simon (1987) found that a situation is not recognizable or retrievable from long-term memory if its form does not match precisely an already existing representation, although this “specificity of access” may be remediated by training. (p. 70) According to Arcavi (2003), recognition is context dependent “... what we see is not only determined by the amount of previous knowledge

which directs our eyes, but in many cases it is also determined by the context within which the observation is made.” (p. 232)

Moreover, students as well as teachers may reject diagrammatic reasoning because it is not precise enough, that is, it lacks the safety of symbolic representation, or a familiar algorithm. As for the latter, fluency and flexibility in translating back and forth between a diagram and its analytic representation, requires proficiency. (Zbiek et al., 2007) Fluency in a given representation does not necessarily imply fluency in another representation. Kendal and Stacey (2001) report, that proficiency in a given representation is sometimes accompanied by a deficiency in another. Developing competence in handling multiple representations is difficult and labor-intensive because it is non-linear and context dependent. (Schoenfeld, Smith and Arcavi, 1993)

The difficulty of assessing individual’s representational schema of a given concept comes from the fact that it may lack the essential elements or contain elements not considered mathematically connected to the concept. Dubinsky (1991) points out “...it is not possible to observe directly any of the subjects’ schemas or their objects and processes. We can only infer them from our observation of individuals who may or may not bring them to bear on problems-situations in which the subjects seeking a solution were trying to understand a phenomenon.” (p. 103)

Engaging with a given task results in reorganization of thinking, therefore, Dubinsky says, any attempt at uncovering an individual’s schema by presenting new tasks, observing the outward displays, and making inferences about the internal actions, processes, objects, and

schemas is futile. What saves us is observing consistent responses across multiple tasks. (ibid.)

DIAGRAMMATIC REASONING: MISCONCEPTIONS

Fischbein (2001) cautions that "... [visual] models may inspire and support correct mathematical inferences with regard to some properties or theorems, but may lead to wrong conclusions with regard to other" (p. 314). He argues that despite the fact that the original mental models of geometry are abstractions, and despite the fact that through instruction abstract notions are introduced gradually and consistently from middle school to high school, "... adolescents and adults continue to think in terms of the figural models and to draw conclusions which may be legitimate in terms of the figural models, but which may lead to incorrect conclusions with regard to the geometrical objects." (ibid.)

Fischbein believes that such misconceptions occur because in early grades geometric objects are presented to students as concrete objects. Students learn about these concrete objects and their concrete properties as if these were the objects of geometrical reasoning. Even though, through the use of metaphors and formalization, abstraction is introduced later on, adolescents and adults revert to thinking about geometric objects as pictures rather than figural concepts for the lack of proper intervention.

The van Hiele Levels of Geometric Thought

When studying geometric knowledge, modern investigators, generally, use a model of geometric understanding proposed by the Dutch action researchers¹⁴ Dina van Hiele-Geldof and Pierre van Hiele in the late 1950's. This model, carrying the van Hieles' name, is now generally accepted as the "industry standard", the most commonly used method of describing and understanding geometric thinking in children and adults. Jaime and Gutierrez (1995) write that "... van Hiele model of mathematical reasoning has become a proved descriptor of the progress of students' reasoning in geometry and is a valid framework for the design of teaching sequences in school geometry" (p. 592).

Freudenthal (1973) writes that Dina van Hiele-Geldof, following in the footsteps of Tatiana Ehrenfest-Afanasjewa, was intensely interested in the way children perceive spatial relations. It is in response to the gradual "trickle down" effect of formalization of mathematics that the van Hieles undertook their work. (p. 402)

Since the van Hieles' discovery, our understanding of the van Hiele model has been significantly broadened and deepened through the contributions of Usiskin (1982), Mayberry (1981, 1983), Fuys, Geddes, & Tischler (1984), Burger & Shaughnessy (1986), Senk (1989); Gutiérrez, Jaime, Fortuny (1991), Mistretta (2000), among many others. The following is a composite sketch of the van Hiele levels of geometric thought in large part based on the work of Marguerite Mason (2002).

¹⁴ Action research, as defined by Wilfred Carr and Stephen Kemmis, is concerned with the following issues: the improvement of teaching practice; the improvement of the understanding of teaching practice; and the improvement of the environment in which teaching practice occurs. Carr, W. & Kemmis, S. (1986). *Becoming Critical: education, knowledge and action research*. Lewes: Falmer Press, London.

There are five van Hiele levels numbered 1 through 5 (Wirzup, 1976); however, the van Hieles originally numbered them from 0 to 4. The van Hiele levels were named by Hoffer (1979). Here is a brief description of each level:

Level 1:	Visualization	Students recognize geometric objects by appearance only. They are able to compare an object to a prototype, i.e., "... this is a circle because it looks like a pancake ...". Recognition is based purely on perception not on object's properties.
Level 2:	Analysis	Students perceive geometric objects as "collections of properties". Although they may be able to recognize properties, students may not be able to recognize relationships between properties. This level is characterized by the lack of understanding of "necessary/sufficient conditions".
Level 3:	Abstraction	Students are able to perceive relationships between geometric objects and between the properties of these objects. Students may be able to create meaningful definitions and they may be able to justify object properties by giving informal arguments. Students demonstrate understanding of logical implications such as "all squares are rectangles" or "not all rectangles are squares". Students are not able to understand the role or the significance of formal deduction.
Level 4:	Deduction	Students are able to construct [<i>sic.</i>]simple proofs; they understand the role and the significance of definitions, axioms, and theorems. Students are able to differentiate between "sufficient" and "necessary" conditions. At this level a proof cannot be "forgotten" because it can be reconstructed.

Level 5:	Rigor	Students at this level perceive geometry as a study of models. They recognize and can discuss the merits of using various methods of proof, for example, indirect proof or proof by contradiction. At this level mathematical arguments may be stripped of their content and argued formalistically. A proof of a counterintuitive result will be accepted if the argument is valid.
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Table 1 – Brief Description of the van Hiele Levels (Mason, 2002)

The intrinsic knowledge attained at the preceding level becomes extrinsic at a current level. Moreover, each level is characterized by a specific (to that level) linguistic symbolism and relational understanding. Hence, a teacher communicating to her student at a level linguistically higher than that which the student has attained is not going to be understood. (van Hiele, 1986)

In 1982 Usiskin published *van Hiele Levels of Achievement in Secondary School Geometry*, a comprehensive study undertaken with a specific purpose “... to test the ability of the van Hiele theory to describe and predict the performance of students in secondary school geometry”. (p. 8) The study addressed a number of issues, among them: the extent to which a van Hiele level can be identified for each student, what it means to attain a particular van Hiele level, the capacity of the model to predict geometric performance, and the ability of people on different van Hiele levels to understand each other’s communication about geometry.

The study demonstrated that the highest level identified by van Hiele either does not exist or it is not testable. (p. 79) One of the findings showed that van Hiele level 3 is a “guidepost”. At or above level 3 “... success in proof is likely, but below [this level]

failure is just as likely”. (p. 51) Usiskin argues that results show that many students are unsuccessful in geometry and the key factor is the lack of pre-requisite knowledge. (p. 52) Even after a year of geometry many students left the course “not versed in basic terminology or geometric ideas”. (p.53) van Hiele model can place students in levels by means of a simple test. ... Van Hiele level is a good descriptor of a concurrent performance in geometry and a reasonably good descriptor of the future performance. ... Questions regarding mathematical systems are answered in virtual random manner”. (p. 89) Moreover, the study did confirm van Hiele levels properties described in Table 2:

Property Name	Property Description
Fixed sequence:	A person cannot be at level N without first attaining level N-1.
Adjacency:	At each level of thought what was intrinsic in the preceding level becomes extrinsic in the current level.
Distinction:	Each level has its own linguistic symbols and its own network of relationships connecting those symbols.
Separation:	Two persons who reason at different levels cannot understand each other.
Attainment:	The learning process which leads to complete understanding at the next higher level has five phases, i.e., inquiry, directed orientation, explanation, free orientation, integration.

Table 2 – Properties of the van Hiele Model

In her dissertation Mayberry argued that to conform to van Hiele model a student should display the following characteristics with respect to his geometric knowledge:

“... He would fail not only to answer correctly but also fail to understand the intent of a question which required thought about his attained level. ... In terms of test results, subject who could be described as a perfect level N for a given concept would respond

correctly to all questions at and below level N, but incorrectly to all questions above level N. The degree to which a subject fitted into this ideal over a wider range of geometric topics would thus be the degree of confirmation of the hierarchy of the van Hiele levels.” (Mayberry, 1981, p. 10)

Pierre van Hiele predicted, and later it was shown, that a student cannot function adequately on a given level unless she has passed through and learned to think intuitively on each of the preceding levels. However, van Hiele believed, a student may be able to “act as if” level-attainment was achieved by performing algorithmically on that level, applying rules that she does not understand and sees them as arbitrary, similar to Skemp’s idea of procedural vs. relational understanding. This property of *level reduction*, which occurs when students have to resort to rote memorization, has been reported by Fuys, Geddes, & Tischler (1988) and Clements & Battista (1992). Fuys et al. describe observing that “... some [students] tried to recall (rather than think out) what their teacher had told them...thus, when geometry was taught, it appeared to be mainly at a recall of knowledge level” (p. 155).

The work of both Mayberry (1981, 1983) and Burger and Shaughnessy (1986) has identified students to be on different levels for different concepts. Vygotsky (1962) writes, “... a concept is more than the sum of certain associative bonds formed by memory, more than a mere mental habit; it is a complex and genuine act of thought that cannot be taught by drilling but can be accomplished only when the child’s mental development itself has reached the requisite level” (p. 82). According to his theory it is necessary for students’ learning that teaching offers *scaffolding* within students’ *zone of proximal development*. Such instruction will eventually enable students to master skills and develop deeper

understanding of concepts. Pegg (1993) writes that the work of Vygotsky regarding the zone of proximal development has to be considered in evaluating van Hiele levels in relation to teaching and curriculum development. In an interview setting "... students' [van Hiele] level is basically what they can offer spontaneously to some stimulus or question." (p.25)

Gutierrez, Jaime & Fortune (1991) found that the degree of acquisition of van Hiele levels is not fixed, but varies across a spectrum. Their findings show that for 46 of the 50 students tested, the degrees of acquisition follow a decreasing order for all four levels, that is, the degree of attainment of Level 1 is greater than that of Level 2, which in turn is greater than the attainment of Level 3, and so on. This result contradicted the belief that van Hiele levels are discrete.

Teacher Knowledge for Teaching

Swafford (1997) writes "... The common belief is that the more a teacher knows about a subject, and the way students learn, the more effective that individual will be in nurturing mathematical understanding". (p. 467) It follows then, that teachers' knowledge for teaching requires closer examination. According to Shulman (1986, 1987) teachers' knowledge for teaching consists of three general domains, namely, pedagogical content knowledge, curricular content knowledge, and subject-matter content knowledge.

The pedagogical content knowledge in which Shulman (1986) includes: "... the most useful forms of representation of [content], the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and

formulating the subject that make it comprehensible to others.” (p. 9) – in this study will be defined as the knowledge of learning theories, teaching theories, pedagogical moves, and educational tools.

The curricular content knowledge may be thought of as the knowledge of standards, scope, sequence, and pacing of the material to be covered.

The subject-matter content knowledge is the foundation upon which the other two domains are constructed. Gardner (2004) defines the knowledge of subject-matter as “... content, formalisms, procedures, [and] dialogic processes ...”. (p. 234) Ball & Bass (2000, 2003) differentiated between the actual mathematical content that teachers use in teaching mathematics and the specialized knowledge of mathematics needed for teaching. Hill, Rowan, and Ball state that such specialized knowledge belongs in the domain of mathematics not pedagogy and it includes, for example, ways of representing arithmetic concepts diagrammatically, or “ ... appraising the mathematical validity of alternative solution methods ...” for a given problem. (Hill et. al., 2005, p.377) Moreover, teachers “... must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students.” (Ball et al, 2008, p.399)

Therefore, subject-matter content knowledge may be further subdivided into two sub-domains, namely the subject-matter content, i.e., in case of Geometry – definitions, postulates, theorems, etc.; and the “subject-matter superstructure” – the knowledge of problems, proofs, historical development, philosophical implications, connections to other areas of mathematics or science, applications, and models. In particular, as it pertains to

Geometry, Chinnappan and Lawson (2005) identify two types of teachers' knowledge, namely, knowledge of Geometry and knowledge of Geometry for teaching.

The quality of teachers' mathematical knowledge for teaching is increasingly the subject of research studies. Ball, Lubienski & Mewborn (2001) examined how teachers know subject-matter content knowledge and "...what they are able to mobilize mathematically in the course of teaching ...". (p. 451) Schoenfeld (1988), Robinson, Even & Tirosh (1992) suggested that the depth of teachers' knowledge for teaching depends on the richness of the interconnected schemas of subject-matter content knowledge. Chinnappan (1998a, 1998b) found that greater linking between geometric and trigonometric knowledge reflects deeper and richer quality of subject matter understandings.

Numerous studies documented that both in-service and pre-service teachers test at low levels of the van Hiele model (Mayberry, 1983; Hershkowitz & Vinner, 1984; Fuys, Geddes, & Tischler, 1985; Mason & Schell, 1988; Swafford, Jones, & Thornton, 1997; Sharp, 2001). Using semi-structured interviews to study the geometric reasoning of 19 pre-service teachers, Mayberry found that 13% of their responses were at a pre-recognition Level 0, 20% were at Level 1, 19% were at Level 2, 24% were at Level 3, and 25% were at Level 4; and there were no responses beyond Level 4. Hershkowitz & Vinner (1984) studied 5th through 8th grade students' and their teachers' geometric knowledge, reporting that both groups had low level of knowledge concerning basic geometrical figures and their attributes; moreover, both groups exhibited similar patterns of misconceptions in geometry. Similarly, Fuys (1985), who interviewed students from 6th through 9th grades and elementary in-service teachers reported similar deficiencies among the two groups. Both

studies conjectured that increasing teachers' geometric content knowledge, specifically, will improve instructional practice. Mason & Schell (1988) combined Mayberry's interview questions and a written protocol from Usiskin (1982) as the data source for their van Hiele-level analysis of 67 pre-service elementary teachers. They found that 38% of the elementary pre-service teachers in their study were functioning below Level 4; and 8% below the lowest level – Recognition.

Chinnappan and Lawson (2000) examined a relationship between successful geometric problem-solving and quality of organization of geometric knowledge. They concluded that the difference between more successful and less successful geometry problem solvers is not the recognition of geometric forms, but their spontaneous abilities to access geometric rules. (p. 39)

Towards Theoretical Framework

The purpose of this study was to improve our understanding of pre-service teachers' geometric knowledge and the tools available to researchers for assessing it.

Regarding research in mathematics education Schoenfeld (1997) writes that the process must begin with an imposition of an interpretive framework and continue with the selection of representation considered relevant to the question at hand. He suggests that one must ask “what is important” within the representational framework and whether the means of analysis are consistent, that is, “... will anyone trained in the analytical methods draw the same conclusions from the same data”. (p. 453)

Geometric knowledge has been described in the literature by many esteemed researchers and although they may have used different terminology their understanding of geometric knowledge is essentially similar. Gray et al (1999) describe two types of cognitive development in elementary mathematics. “One is the van Hiele development of geometric objects and their properties from physical perceptions to platonic geometric objects. The other is the development of symbols as process and concept in arithmetic, algebra and symbolic calculus”. (p. 116) The authors go on to say that one focuses on the *properties of objects* while the other focuses on the *properties of processes*. (p. 118)

Geometric knowledge is representational. A representation is an encoded construct with its own syntax and semantic. (Larkin & Simon, 1987) Representations may be of four types visual, numerical, verbal (Tall, 1991) and physical (Zandieh, 1997; Berry & Nyman, 2003).

Gray and Tall (2004) conceived a theory of three worlds of mathematics. They explain that in addition to three distinct types of mathematical concept, as it pertains to geometry (geometric, symbolic, and axiomatic); there are three distinct types of cognitive development, hence the three worlds. The first, the embodied world stems from our senses. As our verbal abilities evolve and we are able to better describe our environment so grows the embodied understanding. The embodied world is the conceptual world. The second is the world of “*procepts*” – those, which at the same time are *processes* and *concepts* – i.e., the symbolic world. Finally, there is a formal world that which is axiomatic, the world of advanced mathematical thinking. This view essentially supports the van Hiele model. The worlds can be attained through the development of the following “... recognition (of similarities, differences and patterns) that may be categorized to give new thinkable

concepts repetition (of sequences of actions) that may be routinized into automatic procedures ..., [and] ... language, that enables categorization of thinkable concepts, encapsulation of actions as symbols that can act flexibly as process to do or concept to think about and definition of mathematical structures in a formal sense.” (Tall, 2008) For example, the introduction of definitions in geometry and the 'rules of arithmetic' in algebra' can lead to formal embodiment in geometry and formal symbolism in arithmetic and algebra.

Fischbein (1993) defines a geometrical figure by three descriptors, namely: the definition, the image, and the figural concept, (reminiscent of Peircian triad). A geometrical figure is a mental image “completely controlled by a definition” (i.e., a circle – the meaning of the word “is not reducible to a formal definition”); the drawing of a geometric figure is not the figure itself, but a “graphical embodiment” of it; and that the mental image of a geometric figure is a representation of a model, while the geometric figure itself is an abstract idea, a “figural entity strictly determined by its definition”.

A similar idea has been expressed by Tall and Vinner (1981). They have introduced the idea of *concept image* and *concept definition*. Concept definition describes the idea of mathematical meaning while concept image means “the total cognitive structure that is associated with a concept which includes all mental pictures and associated properties and processes. It is built over the years to experiences of all kinds changing as the individual meets new stimuli and matures” (Tall, 1991, p.7) Fischbein (1993) writes that in geometry “... the ideal figural concept corresponds with a concept definition while its mental

reflection with all its connections and ambiguities corresponds with what Tall and Vinner call the concept image.” (p. 143)

A concept image may be characterized as a collection of knowledge schemas. Chinnappan (1998) proposes thinking of geometric schemas as a key concept that anchor other concepts and that geometric schemas, composed of organized concepts, principles, and procedures evolve around a particular shape. (p. 203) For example, isosceles triangle schema may consist of sub-schemas about ruler-and-compass construction, base angles, symmetry, altitude, median, angle bisector, area, etc. Koedinger and Anderson (1990) identify *whole-statement* and *part-statement* attributes of a schema, where the *whole-statement* consists of *part-statements*, for example, “... the CONGRUENT-TRIANGLES-SHARED-SIDE schema refers to the two triangles involved while the *part-statements* refer to the corresponding sides and angles of these triangles.” (p. 518)

Koedinger and Anderson (1990) have developed the Diagram Configuration model of geometric knowledge. They found that expert geometry-problem solvers employ a more abstract strategy than novices in activating their knowledge base. Experts’ geometric knowledge is organized in clusters, which the researchers called “perceptual chunks”. These “perceptual chunks” or “diagram configurations cue relevant schematic knowledge” when experts interpret geometric diagrams “... experts have their knowledge organized according to diagrammatic schemas ... These are clusters of geometry facts that are associated with a single prototypical geometric image.” (p. 518) Such organization allows experts to “skip-step” through the solution focusing on relevant structures first. Moreover, another difference between experts and novices noted by

Koedinger and Anderson was “...the enhanced memory of experts for problem-state displays” (p. 542), that is, the authors hypothesized that expert geometric-problem solvers successfully parse diagrams because of the variety of spatial configurations stored in their memory.

According to the Diagram Configuration Model, authors argue, geometric problem solving consists of three major processes without an imposed hierarchy. There is diagram parsing during which the problem solvers recognize familiar diagrammatic configurations and instantiate corresponding schemas. The objects of recognition are simple geometric facts and conjectures based on the diagrammatic configurations. “The final result of diagram parsing is a network of instantiated schemas and *part-statements*.” (p. 522)

During the statement encoding process problem solvers comprehend what is given and encode it in canonical representation in *part-statements*. During the schema search problem solvers recursively apply schemas with the purpose of establishing a link between what is given and what is to be found. (p. 520)

Chinnappan (1998) argues that a geometric schema has two important characteristics, namely organization and spread, where the former refers to the establishment of the connections between ideas the latter refers to the extent of these connections. (p. 203) In this study Chinnappan analyzed geometry problem solving approaches of high- and low-achieving students. He found that both high-achieving and low-achieving students construct solutions by accessing and using geometric schemas; however, the high-achieving group “ ... show the tendency to analyze the problem methodically by adopting a clear path ... “, while the low-achieving group demonstrated “ ... a lack of direction in the

way they went about tackling the problem.” (p. 212) Moreover, the two groups differed in the way they activated and used geometric schemas. The author writes that “The high-achieving students attended to the structural features of the problem and were thus able to form meaningful integrated mental representations which showed the link between the givens and the problem goal.” (p.213) Once such relationship was established, Chinnappan argues, students were able to select from memory *relevant geometric schemas*. In contrast, the low-achieving group focused on superficial elements and could not establish the connections between the given and their own geometric schemas. Schoenfeld (1988) emphasized that knowledge connectedness is a required characteristic of mathematical thinking. Robinson, Even and Tirosh (1992) argued that to understand the depth of teachers’ knowledge it is necessary to examine the network of interconnected schemas and procedures that form the knowledge base. Anderson (2000) writes that the utility of a knowledge structure in problem solving depends on how elaborate are the connections of this knowledge structure.

First Koedinger and Anderson (1990) and later Chinnappan and Lawson (2005) used concept mapping for representing “...the complexity of geometric knowledge base in a manner that focuses on the state of organisation [*sic*] of that knowledge”. (p. 202)

In summary the literature reviewed in the preceding pages revealed the following.

- Mathematics instruction is rooted in constructivist theory thus teachers use a variety of tools for mathematical discovery, concept building, and communication; one such tool is a diagram. Charles Saunders Peirce argued the

- diagrams are essential to mathematics and there's no mathematical reasoning that is not diagrammatic.
- Throughout history mathematicians used diagrams to illustrate and explain mathematical ideas. Intentionally omitting diagrams from mathematical research is a modern idea. It arose as a result of the development of the foundations of mathematics and the formalization of mathematics. New ways of doing mathematics affected the teaching of mathematics. The philosophical differences between the “old ways” and in “new ways” of doing mathematics influenced philosophy of teaching of mathematics. Reuben Hersch argues that neither intuitionism nor formalism take into account the “teachability” of mathematics. Hans Freudenthal was one of the harshest critics of the formalist movement in mathematics education specifically in the case of geometry.
 - There is a new interest among researchers in the use of diagrammatic reasoning when doing and teaching mathematics because of the use of computers in mathematical research and mathematics education. Arcavi defined visualization as a process and product of creation interpretation and reflection upon pictures images and diagrams. Diagrammatic reasoning means to create or observe a graphically rendered cognitive construct to perceive its components and inherent structures to reflect upon these perceptions to intuitively generate new hypothesis and verify them to communicate ideas and finally to make connections.
 - Diagrammatic reasoning is a cognitive tool. A geometric diagram may be thought of as a collection of precepts which in turn consists of sign, an object, and an interpretant, when accompanied by a linguistic representation forms a concept.

When engaged in diagrammatic reasoning learners are involved in refinement of conceptual schemas. Diagrammatic reasoning may be best described by a metaphor of “scaffolding”. When engaged in diagrammatic reasoning learners may encounter certain difficulties, among them high cognitive demand and flexibility of translation between visual and analytic representations. Visual inspection may be hindered by attention to irrelevant details; learning modality may be a factor. Recognition of concepts rendered graphically is context dependent, determined by previous knowledge and by the context in which information is observed; this may be remedied by training. Diagrammatic reasoning may be rejected by students or their teachers because it lacks symbolic representation or a familiar algorithm. Some learners develop misconceptions when reasoning from diagrams – they may think about geometric objects as pictures.

- Research identified two types of logical thinking, namely, deductive/verbal and inductive/intuitive. Inductive intuitive supports sequential deductive analysis and it is involved in geometry problem solving.
- When studying geometric knowledge modern investigators use van Hiele model of geometric thought. van Hiele model describes five levels of geometric thought development achieved through instruction, not through natural maturation. van Hiele model has certain properties described in the literature: fixed sequence, adjacency, distinction, separation, and attainment. This model was thoroughly studied and several tests were developed to test the model, among them one developed by Professor Usiskin.

- Research shows that both in-service and pre-service mathematics teachers test at low levels of the van Hiele model. Teacher knowledge for teaching consists of specific types of knowledge. Richer pedagogical content knowledge is correlated with better geometry problem-solving.

CHAPTER 3 – METHODS & PROCEDURES

This chapter provides a description of the methodology for collecting and analyzing data. It opens with the restatement of the research questions followed by three sections describing study participants, data collection procedures and instrumentation, and data analysis methodology.

This study seeks to explore a possible relationship between geometric knowledge of pre-service secondary mathematics teachers and their visual reasoning skills. More specifically, this study is seeking answers to the following research questions:

1. Is there a relationship between visual-spatial skills, van Hiele levels of geometric thought, and academic experience in Geometry among pre-service secondary mathematics teachers?
2. Is there a relationship between van Hiele levels and diagrammatic reasoning skills of pre-service high school mathematics teachers as they prove/explain visual theorems?
3. Does van Hiele model adequately predict pre-service high school mathematics teachers' knowledge of geometry?
4. Does van Hiele model adequately predict pre-service high school mathematics teachers' knowledge of geometry for teaching?

Participants

This study involved twelve pre-service mathematics teachers in the last semester of their Master of Education and Teacher Certification Program (Intermediate/Senior division) at a

major metropolitan university. The group consisted of four men and eight women. Three of the twelve participants graduated from high school and obtained their undergraduate degrees outside of North America; however, their mastery of the English language was sufficient for graduate studies in mathematics education. Five of the remaining ten had second language proficiency. Two of the twelve participants earned a graduate degree: one in Education from an institution outside of North America, the other in Mathematics from an American university. One of the twelve participants had a professional degree. Only one of the participants had prior professional teaching experience. The remaining eleven either worked as TA's, peer mentors, or tutors during their college years. All participants had Practicum teaching experience as part of their preparation for teacher certification. Eight of the twelve participants majored in Mathematics, with a minor either in Education, Physics, or Computer Science. One of the twelve participants majored in Spanish and minored in Mathematics. One participant majored in English with some undergraduate coursework in Mathematics but not enough for it to be considered a minor. The remaining two majored in Education with a minor in Mathematics. All participants intended to teach upon graduation. One of the twelve participants intended to teach in the elementary division while the rest intended to teach either in the intermediate or senior division. Four expressed the desire to teach college-level mathematics. To protect participants' privacy they were asked to choose pseudonyms and were referred in this study by their chosen names. Participants were compensated financially for the time spent participating in this study.

Methodology of the Study

This study employs a mixed methodology design, specifically, a modified version of the Convergent Design – Transformation Model approach (Creswell, Plano Clark, 2010) to

collect and analyze data. Convergent Design method was previously mentioned in the literature as the Triangulation method. (Moore, 1991, Tashakkori & Teddlie, 1998, Creswell, et al., 2003)

The purpose of this model is to obtain different but complimentary types of data for analyzing the same topic. (Moore, p. 122) The traditional model of this design prescribes collection of quantitative and qualitative data and then transforming one of the sets either to a qualitative or a quantitative type. Such methodology allows a researcher to directly compare quantitative results with qualitative findings, to validate, or augment quantitative results with qualitative data. (Creswell, Plano Clark, 2003). The diagram below (Figure 4) gives an overview of the model:

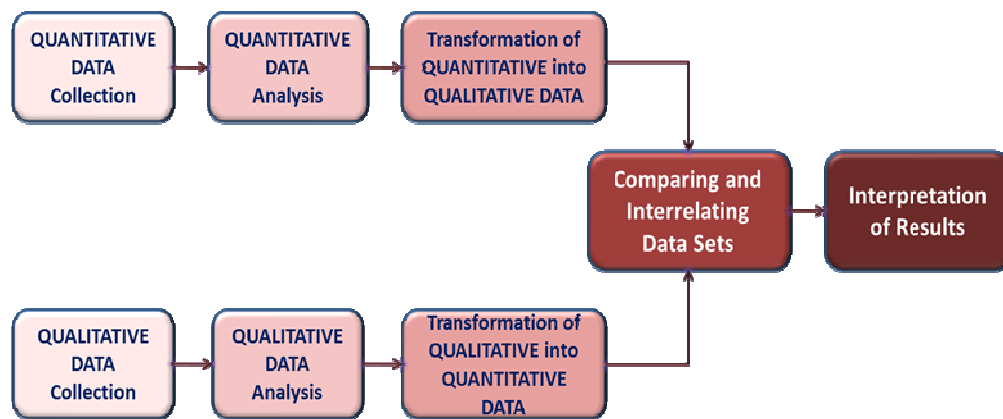


Figure 5 – Methodology Model

According to Creswell et al., this type of design has strengths and challenges. Among its strengths the following should be mentioned: the design is intuitive, the data collection is efficient; and the methodology is flexible. A major challenge of the convergent methodology is that once both sets of data are transformed the results may not agree or support each other, in which case Creswell (ibid.) suggest collecting additional data. All

data sets for this study were collected during interview sessions but analyzed after the data-collection phase. However, in the case of this study, disagreement in data sets would be considered as a meaningful result. The rationale for using this design methodology is that collected quantitative data provides a baseline against which qualitative data collected through interviews may be analyzed.

Data Collection Procedures

Data was collected over a period of three weeks. The investigator met with each participant privately in a study room of the university library for an interview. An interview consisted of two parts.

Part I (60 minutes): In this part of the study each participant completed a CV Questionnaire, modified Purdue Spatial Visualization Test, and a modified van Hiele Levels Geometry Test, all offered online using SurveyMonkey.com functionality.

Part II (120 minutes): In this part of the study each participant was engaged in a *clinical interview*. During the interview each participant was shown two sets of five PowerPoint™ slides containing “proofs without words” (Nelsen, 1997) and asked to prove/explain theorems represented by diagrams. Each participant was given a “hard copy” of the slides to write down notes, perform computations, or draw while “thinking aloud”. The interview was audio-recorded.

Data Analysis

Upon completion of the data-collection phase, tests were scored and the scores were evaluated. The interview recordings were transcribed. Three faculty members from the

Department of Mathematics of the college where the researcher works were recruited to prove/explain visual theorems used in this study with the purpose of creating *concept maps*. Based on their solutions *composite concept maps* were developed for each visual theorem. Concept maps were created based on participants' interview responses and compared to the experts' composite concept maps. Data obtained with all instruments was analyzed and interpreted to answer research question.

Instruments

Four types of instruments were used: a) Curriculum Vitae, b) Visual-spatial ability test, c) Van Hiele levels of geometric thought test, d) Interviews.

CURRICULUM VITAE

Participants were asked to provide information regarding their academic background, their knowledge of English language, and their teaching experience. According to Hill and Ball some studies show that teachers' academic experience (i.e., mathematics courses taken) is a good indicator of what teachers may actually know and "... this indicator is somewhat more consistent in showing effects on students' achievement..." (Hill & Ball, 2009, p. 69) Additionally, *raison d'être* for collecting information about participants' prior academic experience has to do with the properties of the van Hiele model, namely, the significance of language development and prior geometric experience. More specifically, in the case of the latter, the key element of the *attainment property* is that geometric understanding depends on prior instruction more so than on age (Crowley, 1987).

PURDUE SPATIAL VISUALIZATION TEST

To measure participants' spatial abilities *Purdue Spatial Visualization Test* (PSVT) (Guay, 1977) was administered. This test was chosen because in comparison with other spatial ability tests when taking PSVT subjects are least likely to use their analytic abilities to answer the questions. (Bodner & Guay, 1997, p. 14)

Originally developed by Roland Guay and Ernest McDaniel for testing spatial ability of science students, this instrument consisted of thirty test-items of increasing difficulty. Initial items require a rotation of 90° on one axis followed by items requiring 180° rotation about one axis, these followed by rotations of 90° about two axes, and concluding with items requiring rotation of 90° about one axis and 180° about another axis. Since then, the test was modified. (Bodner & Guay, 1997)

To conserve time the PSVT test administered in this study was a modified version of the original 36-item test. This modified version consisted of three parts just as the original version did; however, each part contained only eight questions, twenty-four in total. Research shows that the modified version of PSVT has high construct validity in measuring visual-spatial ability. Guay (1980) reports results of several studies where the internal consistency coefficient, Kuder-Richardson₂₀ (KR-20) was consistently high, (i.e., between 0.87 and 0.92). Battista, Wheatley and Talsma (1982) administered the PSVT to 82 pre-service elementary teachers enrolled in an undergraduate geometry course, and reported the internal consistency coefficient $KR-20 = 0.80$. Bodner and Guay (1997) reported that KR-20 computed based on the test results obtained from 4800 students

enrolled in freshmen and sophomore chemistry courses at Purdue University confirmed internal consistency of the modified version. (p. 9)

The original test was conceived as a paper-and-pencil test, but in this study participants were asked to view test items online and answer questions by selecting a multiple choice response with a mouse-click. The test was administered over a 20-minute period. Each part of the test started with an example and an explanation of what was required of the participant in terms of answering questions that followed. Each example contained its solution. Results were collected and scored.

VAN HIELE GEOMETRY TEST

To assess participants' van Hiele levels of geometric thought this study employed *van Hiele Geometry Test* (Usiskin, 1982). The van Hiele test was designed as part of the Cognitive Development and Achievement in Secondary School Geometry project (ibid.) to test the ability of van Hiele theory to describe and predict performance in geometry. The test has been widely used for both diagnostic and research purposes to test subjects of various ages¹⁵. There was some criticism levied at the test (Crowley, 1990; Wilson, 1990), but it was answered (Usiskin & Senk, 1990). The test consists of twenty-five multiple choice questions. The instrument is divided into five groups each of which contains five questions. Each group of five questions corresponds to a van Hiele level. Questions were constructed by the authors of the study based on the original writings of Diana van Hiele-

¹⁵ Usiskin & Senk (1990) report that since the time their paper was published, over twenty years ago, more than one hundred individuals requested permission to use the test. (p. 1)

Geldof and Pierre van Hiele and then reviewed during the development stage by Pierre van Hiele. (Usiskin, 2011) Scoring was done according to the following criteria:

1. A van Hiele level is considered attained if either “3 out of 5” or “4 out of 5” questions are answered correctly. “3 out of 5” is referred to as a “weak criterion” (WC) while “4 out 5” is referred to as a “strict criterion” (SC)¹⁶.
2. If a participant met the criterion for passing each level up to and including level N and failed to meet the criterion for all levels above, then the participant was assigned to level N; (Fixed Sequence property of the van Hiele Model)
3. If a participant passed a higher level (N+1), but failed to pass the preceding lower level (N) this participant would not be assigned van Hiele level (N+1). This participant would be assigned level according to rule 2. (Level Reduction property) (Usiskin, 1982, pp. 22-26)

Although the test was originally administered as a paper-and-pencil test, participants were not allowed to draw or write to aid their thinking process while answering questions. In this study participants were asked to view test items online and answer questions by selecting a multiple choice response with a mouse-click; however, just as in the original test, participants were not allowed to do any writing or drawing to aid their thinking process. The test was administered over a 30-minute period. Results were collected, scored, and some items were selected for test-item analysis across the entire group.

¹⁶ This type of scoring was used in the original study to minimize Type I or Type II errors. (Usiskin, 1982, p. 24)

CLINICAL INTERVIEW

The interview was designed as a semi-structured exercise 120 minutes in length. There is overwhelming evidence that clinical, task-based interview offers a researcher a view of the subjects' knowledge, problem-solving behaviors, and reasoning (Schoenfeld, 1985, 2002). Goldin (1997) writes that "...clinical interview lends itself well to the qualitative study and description of mathematical learning and problem-solving without the exclusive reliance on counts of correct answers associated with pencil and paper tests." (p. 40)

Clinical interviews may be used for observing mathematical behavior of adults and for drawing inferences based on the observations about problem-solvers' "possible meanings, knowledge structures, cognitive processes, affect, or changes in these in the course of the interview". (ibid.)

However, clinical interviewing does not assure a researcher of certainty regarding "... what transpires in a respondent's mind" as the interviewee responds to questions and that the goal is to "prompt the individual to reveal information that provides clues as to the types of processes mentioned above." (Willis, et al., 1991, p. 3)

Piaget pioneered the "think aloud" method; since then it has evolved into a variety of techniques, including open-ended prompting and structured think-aloud protocols (Clement, 2000). Willis et al. write that "think aloud" and "verbal probing" are two most commonly used methods of cognitive interviewing. Both have certain strengths and certain weaknesses. The "think aloud" methodology relies on the study participants' abilities to think aloud or to verbalize as they respond to questions. The technique requires minimal

intervention from the researcher, and therefore, prevents any of the interviewer-imposed bias; however, in order to maximize the results some training of participants is required. Willis suggests a rehearsal before the beginning of the interview to help participants learn to verbalize. (ibid., p. 4) “Verbal probing” is prone to researcher bias and may interfere with the participant’s solution; however, it offers greater interview control. (ibid., p. 7) With respect to this study “verbal probing” may be necessary as a “scaffolding” device.

This study attempted to discover pre-service secondary mathematics teachers’ geometric knowledge base, which, according to Harel (1993) includes: i) ways of understanding mathematical content, i.e., ways of producing meaning or interpretation for a given term, statement, or problem; ii) modes of thinking that govern ways of understanding, problem-solving approaches, and proof schemes; and iii) knowledge of pedagogy, which in turn includes insight on how to pose problems.

This study used problem solving as a vehicle for examining geometric knowledge of pre-service secondary mathematics teachers. A number of studies have used problem solving skills of the study participants to assess mathematical knowledge or understanding (Even, 1993; Baturo & Nason, 1996; Menon, 1998; von Mindon, Walls & Nardi, 1998; Bryan, 1999; Kinach, 2002b; Hiebert & Wearne, 2003; Ben-Chaim et al., 2007), just to name a few. However, a detailed discussion of research on problem solving is beyond the scope of this study

During the clinical interviews participants were presented ten slides each containing a diagram. These were selected from a collection of problems published in “Proofs without

Words” (Nelsen, 1998). For convenience of reference in this manuscript each of the problems was assigned a unique number consisting of a prefix PWW and a number from 1 through 10 (see Appendix – Proofs without Words).

Participants were asked to view these diagrams and answer questions about the diagrams. The investigator followed a script, (see Appendix – Clinical Interview Script), to analyze participants’ responses. Participants were given PowerPoint™ printouts so that they may be able to make notes or write solutions, etc. these notes were collected at the end of the interviews. Participants’ responses were also audio-taped.

Visual Theorems

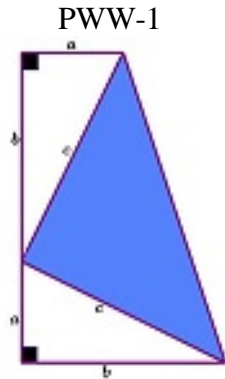
Visual theorems are diagrams, which illustrate a given mathematical result with a minimal degree of ambiguity. Davis (1993) describes three types of visual theorems:

“[1] ... All the results of elementary plane and solid geometry that appear to be intuitively obvious.

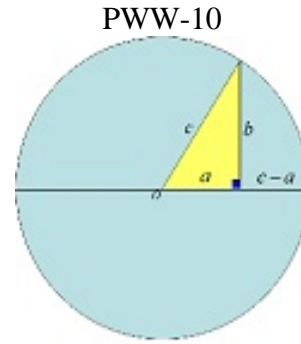
[2] All the theorems of calculus (or of the higher mathematical disciplines) that have an intuitively geometric or visual basis.

[3] All graphical displays (hand-drawn or otherwise) from which certain pure or applied mathematical conclusions can be derived almost by inspection. ...” (p. 336)

Ten visual theorems were selected from “Proofs without Words” by R. Nelsen. Two of the theorems were problems from plane geometry on the Pythagorean Theorem and may be classified as Davis’ type [1] problems.



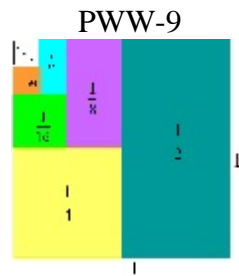
Uses the area of the three triangles and the given trapezoid to verify the Pythagorean Theorem



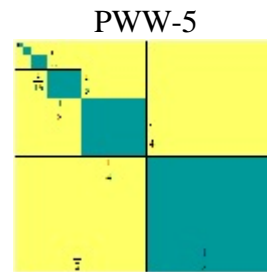
Requires additional construction. Uses similarity of the resulting triangles to verify the Pythagorean Theorem

Figure 6 – Example of visual theorems from plane Geometry

Two visual theorems demonstrated geometric series and may be classified as Davis' type [2] problems.



Sum of areas of half of a unit square, quarter of this square, its eighth, and so on.



Sum of areas of a quarter of a unit square, its sixteenth, etc.

Figure 7 – Examples of visual theorems from Calculus

The rest of the ten problems two from each area of algebra, number theory, and trigonometry may be characterized as intuitively obvious and required only an interpretation of the diagram to obtain a solution, i.e., Davis' type [3]. All ten questions reflected topics suggested by the NCTM curricular guidelines.

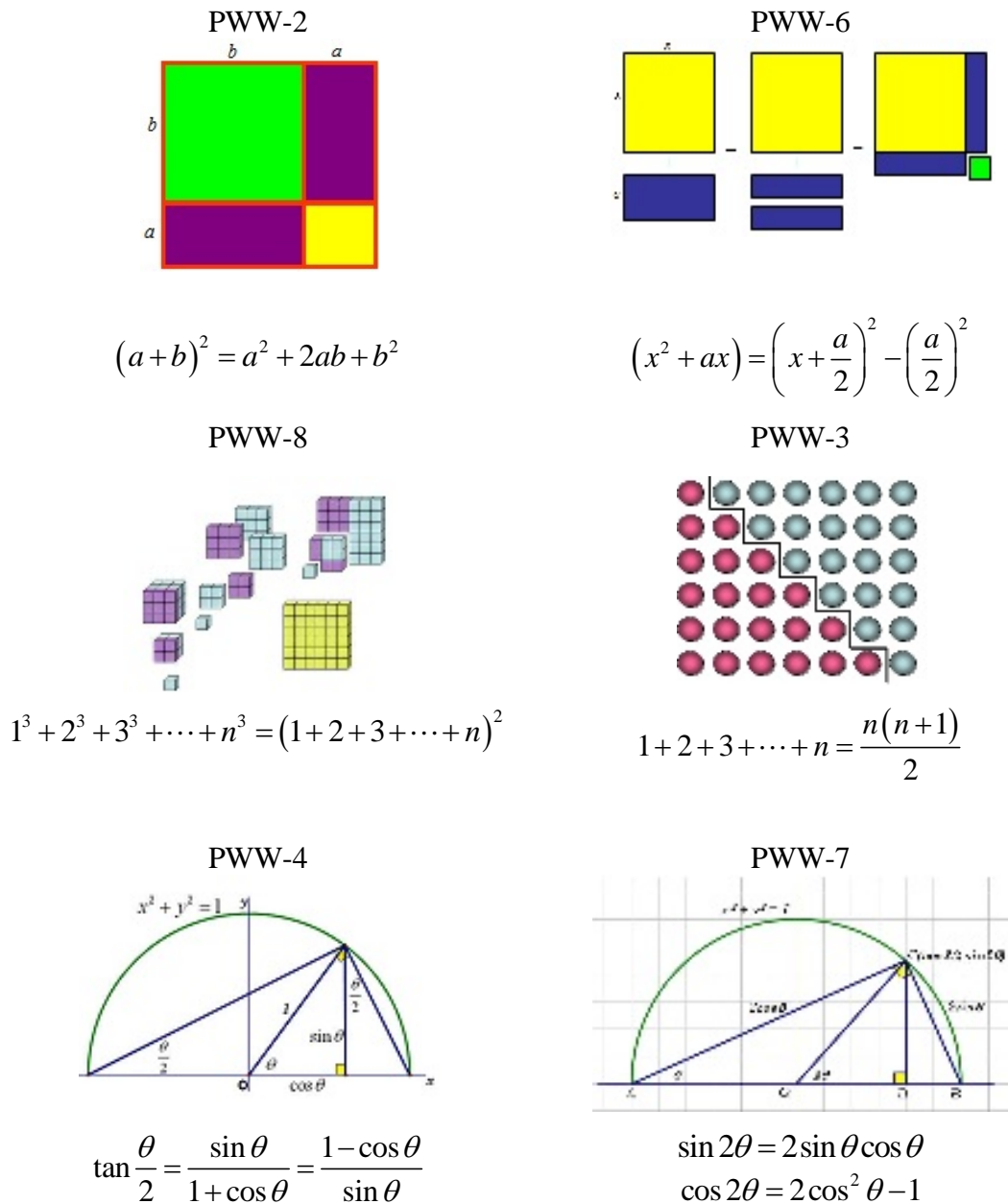


Figure 8 – Examples of visual theorems from Algebra, Number Theory, Trigonometry

CONCEPT MAPS

A number of studies in mathematics education used concept mapping to identify subjects' mathematical knowledge, or misconceptions. (Feldsine, 1983; Mansfield & Happs, 1989; Al-Kunifed & Wandersee, 1990; Hasemann & Mansfield, 1995; Chinnappan & Lawson,

2000, 2005) This method of analysis is well suited for studies investigating issues related to secondary or post secondary education. (Hasemann & Mansfield, p. 47)

Williams (1998) used concept maps to access students' conceptual knowledge of functions. She prepared expert conceptual maps by recruiting mathematicians and questioning them about functions; afterwards she compared their conceptual maps with those of her subjects.

Chinnappan and Lawson (2005) used concept maps in the analysis of teachers' geometric content knowledge. They characterize method of data collection known as concept mapping to be "An intuitively appealing and effective procedure for representing [teachers'] knowledge structure ..." (p. 202)

To obtain data regarding participants' ability to read the diagram, to interpret it and to explain the intent of the visual theorem this investigator recruited, from among her colleagues, three mathematicians – "experts". These experts were asked to prove/explain the same PWW's as were presented to the study participants during the clinical interviews. Just as the study participants, the experts were asked to "think aloud" while working on PWW's. Based on their notes and commentary a composite *Solution Map* was developed for each problem. Each composite solution map was intended to serve only as a guideline or a point of departure for assessing study participants' explanations since there are various ways of obtaining any given proof. Mapping the experts' explanations of PWW's offered an opportunity to examine and analyze the process by which these explanations/proofs were developed.

In their (2005) study Chinnappan and Lawson organized concept maps along two vectors, namely, the range of knowledge and the depth of elaboration. The former related to geometric knowledge and the latter to geometric knowledge for teaching. They write that “The scope, or range of knowledge, can be seen as a quantitative feature of a knowledge base that reflects teachers’ knowledge of geometry (KG). ... The depth of elaboration is used as a measure of teachers’ knowledge of geometry for teaching (KGT) because we contend that the elaborations constrain teachers to reflect on their content knowledge and deconstruct them in ways that their students could relate to.” (p. 209)

First Koedinger and Anderson (1990) and later Chinnappan and Lawson (2005) used concept mapping for representing “...the complexity of geometric knowledge base in a manner that focuses on the state of organisation [*sic*] of that knowledge”. (p. 202) In this study the method used for constructing solution maps is an amalgam of two methods that of Chinnappan and Lawson (2005) and Koedinger and Anderson (1990). The solution maps are organized by means of nodes and connections as in Chinnappan and Lawson, where nodes signify whole-statements and connections signify part-statements as in Koedinger and Anderson.

CHAPTER 4 – DATA ANALYSIS AND FINDINGS

In this chapter the reader will find analysis of data and discussion of findings obtained in this study. The chapter opens with the analysis of data obtained through questionnaire and testing instruments followed by the analysis of solution maps of experts' and study participants.

Geometry Score

According to the van Hiele model, advanced geometric understanding depends on instruction in Geometry. Consequently, the following indicators in the participants' Curriculum Vitae were considered significant:

Geometry learning in high school: Participants were asked whether they studied Geometry as a separate course or whether Geometry was part of the Integrated Mathematics (i.e., Regents Mathematics A, B). A separate course was assigned a value of **1** otherwise the answer was assigned a value of **0**. This item was considered significant for the reason that studying Geometry as a separate course devotes more time to the subject of Geometry. In the integrated mathematics curriculum items pertaining to Geometry are distributed across curriculum and often do not receive adequate treatment. Recognizing a curricular deficiency, New York State Department of Education mandated the development of mathematics curriculum which includes one year of Geometry as a separate course. Among this study's participants **70%** reported taking Geometry as a separate course.

Participants were asked whether while in high school they studied "Geometry with Trigonometry". During the interview, the investigator had to clarify the difference between

“learning Geometry *and* Trigonometry” as in two separate disciplines or distinct units in the mathematics curriculum and “learning Geometry *with* Trigonometry” as in using trigonometric properties to solve or prove geometric results. The latter signifies a curriculum of greater complexity and depth. (Chinnappan & Lawson, 2005) An affirmative answer was assigned a value of **1** otherwise the answer was assigned a value of **0**. Among this study’s participants **50%** reported studying geometry with trigonometry.

Studying “*ruler and compass*” constructions or “*proving*” geometric results was assigned a value of **1** if in the affirmative otherwise it was assigned a value of **0**. The NCTM Principles and Standards for School Mathematics underscore the importance of teaching “ruler and compass” constructions and proving geometric results. (NCTM, 2005, A, B) Among this study’s participants **42%** reported doing “ruler and compass” constructions or “doing proofs” while in high school.

Responses regarding *the number of Geometry courses* taken in participants’ undergraduate and graduate programs indicate that of the twelve study participants **5 (42%)** took a Geometry course beyond high school.

Responses regarding *the number of courses about spatial relations*, such as Linear Algebra, Topology, Graph Theory, etc., taken in participants’ undergraduate and graduate programs indicate that of the twelve study participants **83%** took upper-level courses about spatial relations beyond high school. On the average, those participants, who majored or minored in Mathematics and not Mathematics Education, took **2.1** upper-level courses about spatial relations.

Study participants were asked whether they took a course in problem solving. An affirmative answer was assigned a value of **1** a negative answer was assigned a value of **0**. Of the twelve study participants **75%** reported taking a formal course in problem solving.

The number of Mathematics Methods courses taken in participants' undergraduate and graduate programs referred to two separate items, namely, a course in Geometry Methods or topics in geometry methods taught in general methods courses in particular and Mathematics Methods in general. None of the participants reported taking a formal course in methods for teaching Geometry. All participants reported taking at least 1 or at most 3 courses in teaching methodology. However, even a unit on methods for teaching geometry was not included in any of the general methods courses. The average number of methods courses taken by each participant is approximately **1.7**.

Items detailed above were summed to obtain the ***Geometry Score (GS)***. Table 3 reflects study findings regarding participants' Geometry Score. 75% of pre-service mathematics teachers in this study learnt geometry as a separate subject; and 75% took a problem solving course. Half of the participants studied geometry with trigonometry, which indicates an enriched curriculum. Only 42% of the participants reported studying "ruler and compass" constructions or doing proofs in high school; and the same percentage reported taking an upper-level course in geometry. Of the twelve study participants, two did not take any other spatial relations courses in their academic career beyond high school. The ten participants that did take spatial relations courses, on the average, took 1.8 courses ranging from one to four courses. None of the participants took geometry methods courses and all of the participants took at least one Mathematics methods course.

Participant's name:	Matthew	Mandy	Timmy	Katie	Barbara	Peter	Ashley	Jenny	Abigail	Polly	Jerry	Nancy	Group Averages
Did you learn Geometry as a separate subject in high school?	0	1	0	1	1	1	1	0	1	0	1	1	75%
Did you learn Geometry with Trigonometry?	1	1	0	0	1	1	1	0	0	0	1	0	50%
Did you study "ruler & compass" constructions or "do proofs"?	0	1	0	0	1	1	0	1	0	0	1	0	42%
How many Geometry courses did you take in college or grad school?	1	0	0	0	0	0	1	0	1	1	1	0	42%
Total number of Topology, Linear Algebra, Graph Theory taken.	2	0	4	1	1	2	2	1	0	1	3	4	1.8
Did you take a course in Problem Solving?	1	0	1	1	1	0	0	1	1	1	1	1	75%
How many Geometry Methods courses did you take?	0	0	0	0	0	0	0	0	0	0	0	0	0
How many Mathematics Methods courses did you take?	1	2	1	2	2	1	2	1	2	3	1	2	1.7
Geometry Score	6	5	6	5	7	6	7	4	5	6	9	8	6.2

Table 3 – Summary of the Geometry Score Computation

Visual Spatial Skills

Spatial ability was measured using the Purdue Spatial Visualization Test (PSVT) (Guay, 1976). The test consists of twenty-four questions, each correct answer was assigned a score of **1** and each incorrect answer was assigned a score of **0**. Thus, the maximum obtainable score was 24 points. The results (see Table 4) indicate that study participants' visual-spatial skills are in the medium to high range with almost **60%** of the participants testing above group average. The results of PSVT were recorded and compared with the results of the van Hiele Geometry Test.

Name	Raw Score	Weighted Score
Matthew	21	88%
Katie	21	88%
Jerry	21	88%
Ashley	20	83%
Timmy	19	79%
Jenny	19	79%
Abigail	19	79%
Barbara	17	71%
Nancy	17	71%
Mandy	15	63%
Peter	13	54%
Polly	12	50%
Group Average	17.8	74%

PSVT Raw Scores

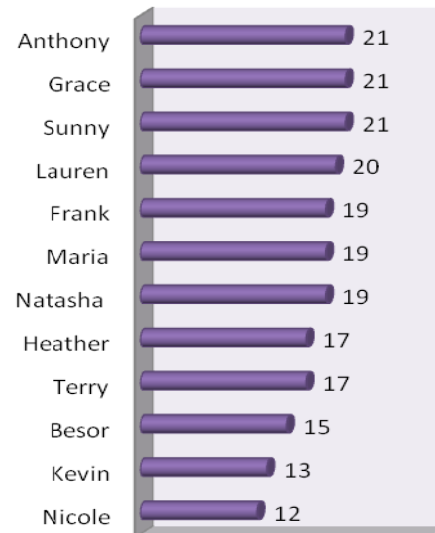


Table 4 – Purdue Spatial Visualization Test Results

van Hiele Levels

In this study the evaluation of participants' van Hiele levels was done based on the method employed by Professor Usiskin in his 1982 study "van Hiele Levels and Achievement in Secondary School Geometry". The van Hiele Geometry Test (Usiskin, 1982) was scored, data was recorded and analyzed (Appendix – van Hiele Geometry Test Scores). The summary of the data obtained, is shown in the Table 5 below. Data recorded in the table reflects total scores in each level for every participant.

	van Hiele Levels					Weighted Sum		Attained vHL	
	1	2	3	4	5				
	2 ⁰	2 ¹	2 ²	2 ³	2 ⁴				
	1	2	4	8	16	WC	SC	WC	SC
Matthew	5	5	5	3	5	31	7	5	3
Jerry	5	5	5	3	5	31	7	5	3
Nancy	5	5	5	3	5	31	7	5	3
Ashley	5	5	5	1	4	7	7	3	3
Jenny	5	5	5	1	3	7	7	3	3
Abigail	5	5	4	1	5	7	7	3	3
Katie	5	4	4	3	4	31	7	5	3
Timmy	5	4	4	1	4	7	7	3	3
Barbara	5	3	5	2	1	7	1	3	1
Peter	5	3	3	1	2	7	1	3	1
Mandy	5	3	1	2	2	3	1	2	1
Polly	5	2	3	1	2	1	1	1	1

Table 5 – van Hiele Geometry Test Results

Columns, labeled **van Hiele Levels (1 – 5)**, contain raw scores for vHL1 through vHL5. The 5th van Hiele level – Rigor – was included and both criteria for level attainment were used in evaluating results, i.e., the “weaker criterion” (**WC**) – “3 correct answers out of 5 possible correct answers” and the “stricter criterion” (**SC**) – “4 correct answers out of 5 possible correct answers” (Usiskin, 1982, p.23). Since each level consisted of five questions and each correct answer was assigned a value of 1 the lowest possible raw score for a given level was 0 and the highest possible raw score was 5. The columns labeled **Attained vHL** contains numbers which indicate van Hiele level attained by the participant according to the Fixed Sequence property of the van Hiele model, i.e., *a participant is considered to have attained a level if and only if this participant attained the given level*

and all levels below it. (p. 25) Scores recorded in the columns of Table 7 labeled Weighted Score and Weighted Sum were obtained as follows:

- Participants' results were assigned "weights" as in the original study (ibid., p. 22), that is, because of the binary nature of the evaluation – i.e., either the level was attained (**1**) or it was not (**0**) – each level was assigned a numeric value 2^{L-1} , where L is a given van Hiele level.

Van Hiele Level	1	2	3	4	5
2^{L-1}	2^0	2^1	2^2	2^3	2^4
Weights	1	2	4	8	16

Table 6 – van Hiele Levels' Weights

- For example, Peter had the following raw scores: 5, 3, 3, 1, 2. Using **WC** his weighted score is $1+2+4+0+0=7$ while using **SC** his weighted score is $1+0+0+0+0=1$ and so the entry in the table is recorded as **7** in the **Weighted Score WC** column and as **1** in the **Weighted Score SC** column. Similarly, the level attainment with respect to the Fixed Sequence property, i.e., **Weighted Sum**, using **WC** is $1+2+4+0+0=7$, but using **SC** is $1+0+0+0+0=1$ and so the entry is recorded **7** and **1** respectively. Consequently, **Attained vHL WC** contains value of **3** and **Attained vHL SC** contains **1**.

According to the obtained results with respect to the of the van Hiele model, Matthew, Jerry, Nancy, and Katie attained vHL5 based on the weaker criterion (WC). None of the participants attained vHL5 based on the stricter criterion (SC). Barbara, Peter, Mandy, and

Polly attained vHL1 based on the stricter criterion, the rest eight participants attained vHL3 based on the stricter criterion. Table 9 contains percentage of participants who attained a given van Hiele level using **WC** and **SC**.

<i>Van Hiele Level (vHL)</i>	<i>Level Attainment Weaker Criterion (WC)</i>	<i>Level Attainment Stricter Criterion (SC)</i>
1	100%	100%
2	92%	67%
3	83%	67%
4	33%	0%
5	33%	0%

Table 7 – Percent of participants attaining given van Hiele Level

Data in Table 7 indicates that 100% of participants attained vHL1. 92% and 67% attained vHL2 using the **WC** and **SC** respectively (i.e., **WC** = correctly answered at least **3 out of 5** questions and **SC** = correctly answered at least **4 out of 5** questions). 83% attained vHL3 using **WC** and 67% attained vHL3 using **SC**. 33% attained vHL4 and vHL5 using **WC** and none of the participants attained vHL4 or vHL5 using **SC**.

The following Table 8 contains scores of participants across three tests.

	GS	PSVT	vHL's
Jerry	9	21	3
Nancy	8	17	3
Ashley	7	20	3
Barbara	7	17	1
Matthew	6	21	3
Timmy	6	19	3
Peter	6	13	1
Polly	6	12	1
Katie	5	21	3
Abigail	5	19	3
Mandy	5	15	1
Jenny	4	19	3

Table 8 – Summary of Scores

Correlation coefficients were computed using *CORREL* function of Excel™ spreadsheet to obtain the following results:

There appears to be no correlation between participants' visual spatial skills as measured by the Purdue Spatial Visualization Test – PSVT (Guay, 1976) and their geometry scores obtained through questionnaire.

$$r = \text{CORREL}(GS, PSVT) = 0.111, \quad R^2 \approx 0.0124$$

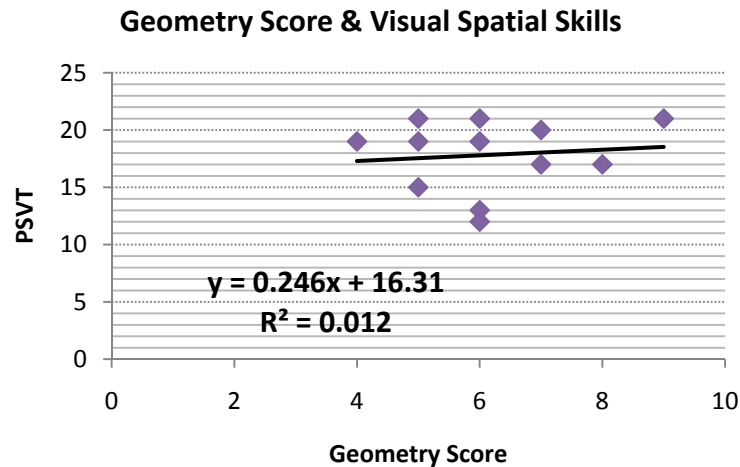


Figure 9 – Relationship between GS and PSVT

Even though the van Hiele model suggests and numerous studies have shown that attainment of van Hiele levels occurs through instruction, there appears to be no correlation between participants' geometry scores (GS) and their attained van Hiele levels (see Figure 10)).

$$r = \text{CORREL}(GS, vHL) = 0.088, \quad R^2 \approx 0.008$$

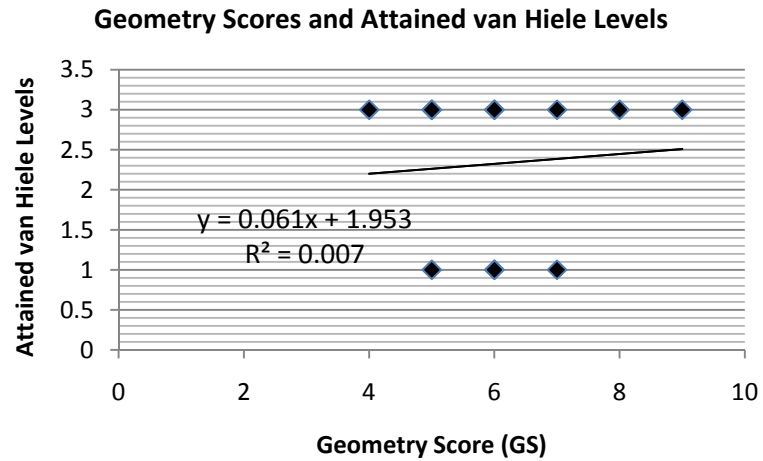


Figure 10 – Relationship between GS and vHL

However, there appears to be a weak correlation between visual spatial skills and van Hiele levels attained by the participants (see Figure 11).

$$r = \text{CORREL}(PSVT, vHL) = 0.854, \quad R^2 \approx 0.729$$

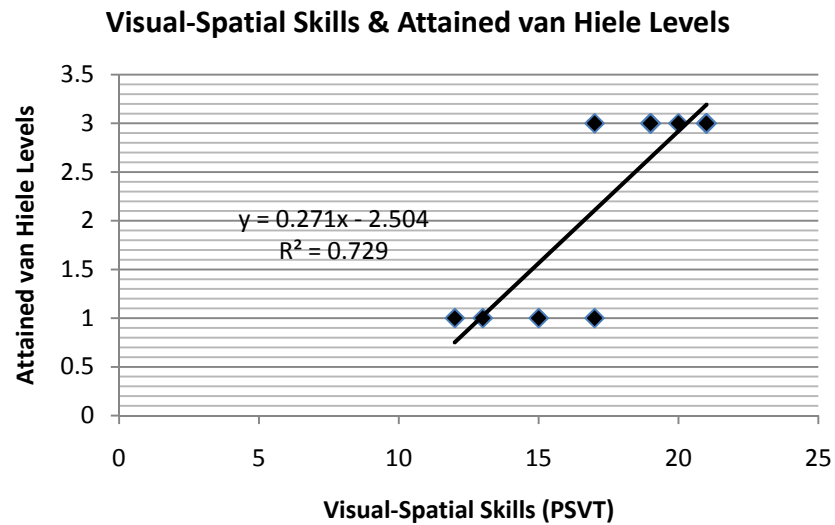


Figure 11 – Relationship between PSVT and vHL

Experts' Concept Maps

Based on the studies of Koedinger and Anderson (1990), Chinnappan (1998), and Chinnappan and Lawson (2005) the following characteristics of experts' strategies for solving geometric-proof problems were considered significant in relation to visual theorems presented in this study:

- Solution sketch – an outline of a solution in general terms, an articulation of a direction in which one proceeds towards a solution.
- Solution space - geometric schemas relevant to the diagrammatic components.
- Step-skipping – intentionally omitting steps in developing a solution based on *schema/sub-schema* constructs.

The process of finding visual explanation/proof generally consisted of three stages: recognizing what the visual theorem is about, sketching explanation/proof plan, and developing a solution.

STAGE 1: RECOGNITION/CONJECTURE

When presented with a visual theorem experts worked on establishing what the given visual theorem was about. The main idea of “*aboutness*”, in this case, the concept image of the given visual theorem, defined the solution space and informed the direction in which the experts proceeded in the next stage. Experts reported observing and recognizing geometric components of the diagram and searching for a relationship between them. In the case of visual theorems from the geometry curriculum (PWW-1 and PWW-10) the experts parsed them into representational schemas similar to Koedinger & Anderson *whole-*

statements schemas, until they were able to recognize or establish the intent of the given theorem. The following graphic (Figure 10) is an example of such decomposition.

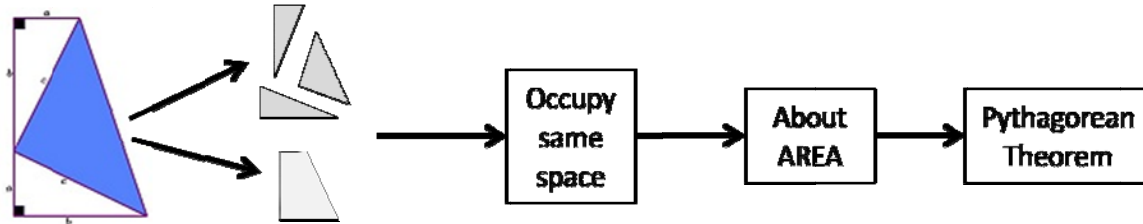


Figure 12 – Experts’ diagram parsing into representational schemas

This recognition was triggered, in the case of PWW-1, by an observation that the given trapezoid was divided into triangles or that the triangles tiled the trapezoid – an embodied concept. The idea of tiling invoked the notion of area, which in turn allowed experts to conjecture that this was a visual proof of the Pythagorean Theorem.

Expert 3 regarding PWW – 1

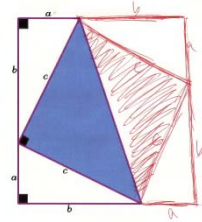
- I: Here is a diagram that represents a theorem. Do you recognize it and if not can you guess what it represents?
- E1: Do I know what it represents? [looks at the diagram]
- I: Just examine it and tell me what you see. Maybe it will come to you.
- E1: So, what do I see? ... I see triangles. What is special about them? They form this trapezoid ... Triangles tile this trapezoid? ... Ah ...they occupy same space [traces the outline of the trapezoid] ... something about area ... I’d say Pythagorean Theorem?

Also, one of the experts recognized that the given diagram was “half” of a well known visual proof of the Pythagorean Theorem, which led him to invoke the Pythagorean Theorem schema.

Expert 2 regarding PWW – 1

I: Do you recognize this diagram, perhaps, you have seen this before? It graphically illustrates a theorem. If you do not know what it is, try to guess.

E3: I have not seen this before, but it reminds me of something ... You know what it reminds me of? ... It reminds me of this ... [draws on the diagram] This diagram I recognize ... this is Pythagorean Theorem.



In case of PWW-10 experts reported that the clue to recognizing the concept image of this visual theorem was the presence of the right triangle and the notation used.

Expert 2 Regarding PWW-10 Pythagorean Theorem

E3: I am just going to guess here. ... I see a circle, a right triangle constructed on the diameter and it is annotated with letters a , b , c ... I suppose they would try to make it obvious by using a mathematical cliché ... Is this another proof of the Pythagorean Theorem?

In parsing the diagram experts relied on both spatial relations and symbolic notation clues to establish the “aboutness” of the visual theorem. From the expert’s statement there is a connection between graphic and symbolic schemas. One of the experts has seen this version (PWW-10) of the Pythagorean Theorem before, and was able to identify it upon presentation.

Eight of the ten visual theorems used in this study, although presented in diagrammatic form, required knowledge of topics in mathematics curriculum other than geometry. Namely, “square of the sum”, “completing the square”, “sum of n integers”, “sum of cubes”, “trigonometric identities”, and “geometric sums”. The diagrammatic rendering of these visual theorems were intuitively obvious to the experts and required no specialized

geometric knowledge beyond “area of a rectangle is $l \times w$ ” construct, or “tiling means area” construct, except for PWW-7, which required the knowledge of similarity property for its solution, but not recognition. In each case experts were able to activate relevant schema of each visual theorem by parsing the diagram. The key factor was the ability to interpret diagrammatic notation and conventions used in diagrammatic communication. For example, a set of dots arranged in a rectangular array (PWW-3) invoked “area of a rectangle” schema even though the arrangement was not in the form of four pair-wise intersecting lines, four right angles, etc.

Expert 2 Regarding PWW-5 Geometric Sum

- I: Do you remember this one?
- E2: Wait a minute ... I do not have to remember anything, I can see it ... this is brilliant [laughs]... a half of side plus a quarter of a side, plus an eighth of a side of a square ... and so on ... cannot be greater than the side of the square, which is equal to one ... there it is ... a geometric sum.

STAGE 2: DEVISING A PLAN

Experts devised a solution plan for each problem given. The solution plan was of general nature, a sketch, confirming Koedinger and Anderson’s (1990) findings, which compared experts and novices geometric problem solving.

Expert 1 regarding PWW - 10

- E1: Based on the problems we did before I bet it has something to do with similarity.
- I: Let’s just follow the process as before.
- E1: OK ... let me see if I can construct something here [draws chords]... then I will look at the resulting triangles. This reminds me of the Trig problem about the double-angle ... I need to establish similarity traces triangles on the diagram]... So, I am going to set up some ratios to see if I can get a result that way.

Expert 3 Regarding PWW-7 The double-angle formula

- I: How are you going to proceed?
 E3: I cannot tell yet; however these two triangles are likely similar and I should be able figure out the ratios of corresponding sides if I need to. ... So, OK ... How do I know they are similar?

The following diagram illustrates the solution plan for PWW-1. When devising a solution plan, experts worked towards a particular goal identified by an informal “formal statement”, a short-hand statement that described the essence of the problem they were working on. For example, $Area_{trapezoid} = \sum Area_{triangles}$

Such statement in itself defined the solution space, was obtained by step-skipping and was the general solution of the visual theorem. Such a plan would allow experts to focus on items that required further parsing. When working on geometric visual proofs the experts employed “from general to specific” method of reasoning, similar to the *whole-statement/part-statement* observed by Koedinger and Anderson (1990).

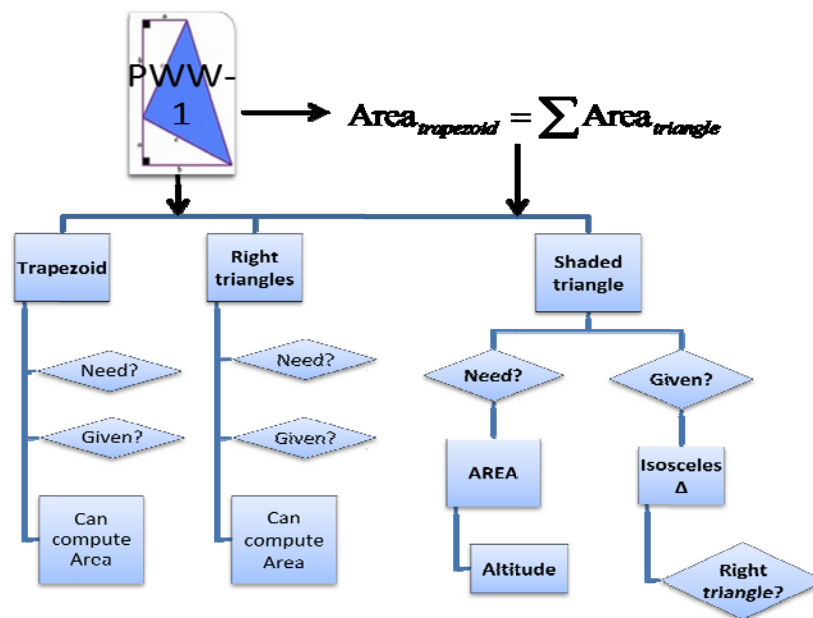


Figure 13 – Experts’ successive refinement of schemas

This diagram illustrates the successive refinement of schemas from general to specific. All three experts looked at the diagram in gestalt, and then went about parsing it, i.e., breaking it down into part-statement components. When considering schema TRAPEZOID two questions were resolved, namely, “what is *needed* to find area?” and “what is *given* to find area?” As these were resolved experts moved to the next schema RIGHT-TRIANGLES and the process was repeated again. When parsing the SHADED-TRIANGLE schema the process is repeated, but now SHADED-TRIANGLE-AREA becomes Stage 1: Recognition/Conjecture schema.

STAGE 3: DEVELOPING A SOLUTION

At this stage of the process experts were ready to organize their solution into a semi-formal argument and begin to use symbolic notation to communicate their solution. Experts’ approach to finding an explanation/proof of PWW’s from geometry curriculum had cyclical parts and it reflected the van Hiele model.

5. Recognition: naming geometric shapes
6. Analysis: identifying properties
7. Pre-Deduction: formalizing arguments
8. Deduction: generalizing results

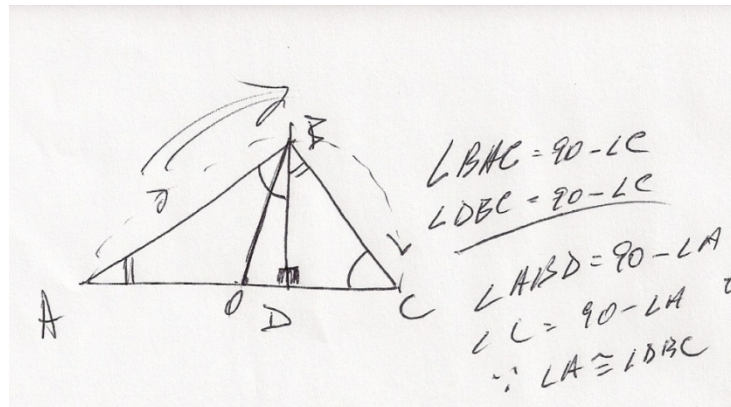
The 5th van Hiele level – Rigor could not be observed due to the nature of the geometric knowledge attributed to this level, not due to the fact that the experts did not attain it. Professor Usiskin reminisced (Usiskin, 2011) about Pierre van Hiele’s review of each test item in the Usiskin (1982) study and his insistence on proper attribution of the levels through adherence to the questions specific to the level.

As in van Hiele Recognition level objects in the diagram of a given visual theorem were identified by naming, for example, “This looks like a trapezoid”, “I see a semicircle”, “It’s a square”, or “It looks like a right angle”; however, since none of the experts qualified their answers by comparing a geometric object (circle, rectangle) to a pancake or a door, but rather relied on definitions or properties of geometric objects it is safe to assume they were functioning at Level 2 – Analysis. The experts were wording their responses using expressions such as: “looks like”, “probably is”, and “seems to me” of which the subtext is that a formal justification will be needed for the formal solution. Also, function of the Analysis is reading conventional geometric notation from the diagram, for example: “...right angle triangle”, “It’s a right trapezoid”, “It’s an isosceles triangle”, or “It’s a unit circle”. Here two types of processes occurred, namely, the information in the diagram (i.e., lengths of sides, coordinates of points, perpendicularity, etc.), given in symbolic form, was read and interpreted and based on this information properties of the given object were identified. Every statement of property was generally followed by a justification, for example, “It’s an isosceles triangle because two sides have the same length – it is given”. Such a statement would be an intersection of two van Hiele levels, namely Analysis and Pre-deduction where properties are not just identified, but some justification is given.

In producing Pre-deduction type arguments, two processes occurred, namely, properties were verified and relationships were established. It is at this stage experts generally asked themselves and answered “How do I know that ...?”. For example:

Expert 2 Regarding PWW-10 Pythagorean Theorem

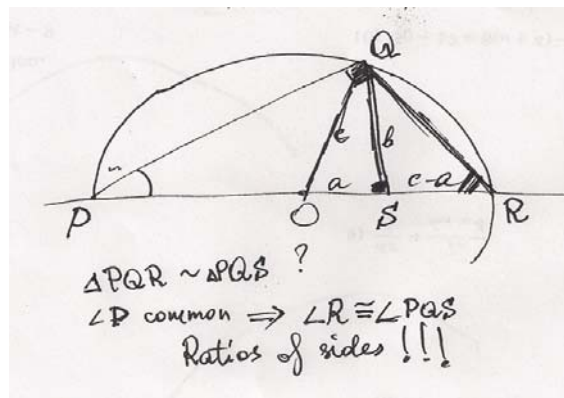
E2: How do I know these triangles are similar? To be similar their angles must be congruent. Are they congruent? ... well ... these are right triangles so we need to focus on these two [points to $\angle A$ and $\angle C$]



It is in this cycle that geometric relationships take on a form of a procept, i.e., a process related to a concept. It describes geometric objects in terms of their properties and an action that follows from these properties.

Expert 3 regarding PWW-10 Pythagorean Theorem

E3: If triangles are similar their sides are in proportion and we can take ratios of sides ... all the needed sides are given.



During this cycle experts relied on their mathematical intuition as in the following example:

Expert 1 Regarding PWW-1 Pythagorean Theorem

I: So then, what is your next step?

E1: There are three triangles, two of them are right triangles, that is given; however, I bet the third one is also right because it seems that the other two angles are complementary ... I have to look closer at that one.

In the course of developing solutions experts generally expressed confidence in being able to either recognize or deduce information needed to proceed with the solution, otherwise they would seek a different approach, which is characteristic of someone functioning at vHL4. For example,

Expert 1 Regarding PWW-5 Geometric Sum

I: So then, what is your next step?

E1: You know, I cannot quite remember the proof It is very simple algebraic manipulation ... I'll just start writing it down ... I am sure it will come to me ... otherwise, I will have to do something else.

Any deduced relationships are given a justification by a statement of a definition, an axiom or a theorem. This is where van Hiele Pre-Deduction intersects the Deduction level when writing a formal solution, experts would organize and logically sequence previously prepared (observed or deduced) geometric sub-schemas paying special attention to consistent notation and logical order of the argument. An alternative solution would be proposed or an insight about the given problem would be shared. For example,

Expert 3 Regarding PWW-10 Pythagorean Theorem

E3: I wonder if to connect only one end of the diameter with the vertex, I mean this one (points to the endpoint farthest away from the base of the perpendicular b), ... I wonder if this is sufficient ... and to use similarity ... a nice question to ask in class ... Can you prove constructing two lines and then constructing only one ... and which one ... Why not the other one? ...

In their analysis of teachers' geometric knowledge for teaching Chinnappan and Lawson (2005) organized concept maps along two vectors, namely, the range of knowledge and the depth of elaboration. The former related to geometric knowledge and the latter to geometric knowledge for teaching. They wrote that "The scope, or range of knowledge, can be seen as a quantitative feature of a knowledge base that reflects teachers' knowledge of geometry (KG). ... The depth of elaboration is used as a measure of teachers' knowledge of geometry for teaching (KGT) because we contend that the elaborations constrain teachers to reflect on their content knowledge and deconstruct them in ways that their students could relate to." (p. 209)

Nevertheless, when a given solution was formalized experts did offer suggestions regarding the use of visual proofs in the mathematics curriculum at the collegiate level as well as connecting to other mathematical results or trying for a more elegant solution – a demonstration of the quality of their geometric knowledge for teaching.

Case Studies

BARBARA was educated in the United States. Her first language was English; however, she was also fluent in Mandarin and French. Upon graduation from the teacher certification program Barbara intended to teach mathematics at the intermediate level. She had a bachelor's degree in mathematics with a minor in English literature and a master's degree in Art. Barbara had little teaching experience – only that, which Practicum offered. By her own admission, Barbara struggled in middle school and high school mathematics, but in her 12th grade she had “... an amazing mathematics teacher who made everything so clear ...” that she had the courage to select mathematics as her undergraduate major. While in high school she mostly studied algebra; however she studied geometry as a separate course, but no trigonometry. The geometry curriculum delivered in her school did contain “ruler and compass” constructions; however, according to Barbara “...I could never understand the point of it ... it seemed so artificial so I did not pay attention”. In her undergraduate program Barbara took a linear algebra course, but never any geometry. Her significant encounter with geometry was in her teacher certification program in a problem solving course. Barbara's GS = 7, PSVT = 17(71%), and she attained vHL1.

Since this study involved pre-service mathematics teachers rather than high school students, the investigator chose to accept the stricter criterion (SC) for van Hiele levels attainment and therefore, since Barbara scored (5, 3, 5, 2, 1) respectively on the van Hiele Geometry Test her attainment level was vHL1. She had difficulties with properties of figures (vHL2), distinguishing between a proposition and its converse, and necessary and sufficient conditions.

During the clinical interview Barbara was able to solve most of the presented visual theorems. When shown PWW-1 she recognized the diagram as “half” of another visual proof of the Pythagorean Theorem.

I: Have you seen this diagram before?

Barbara: No, it does not look familiar

I: What do you think it represents?

Barbara: Uhm ... I am not sure ... Maybe something about the Pythagorean Theorem? ... I see right triangles and ... two triangles the same length sides. An isosceles triangle with side c ... Uhm, let's see. I guess you could say that ... aah you know I think I did see something like this recently similar where you prove Pythagorean Theorem using a square.

Barbara quickly drew the other “half” of the trapezoid, stated her intention to show that the area of the square with side c is equal to the area of the square with side $(a+b)$ without the sum of the areas of the four triangles with sides a, b ; and she produced a sketch of the solution. (Figure 14)

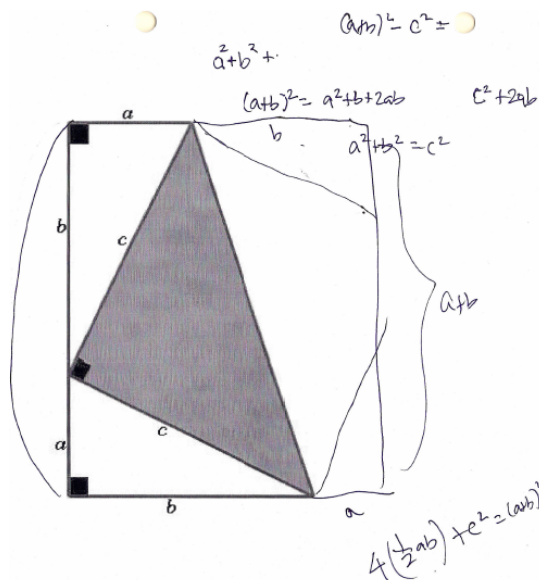


Figure 14 – Barbara’s work on PWW-1

Barbara mentioned that one of her students shared this proof with the class and she remembered it because it was “so neat”. She was also successful in solving PWW-10.

I: Have you seen this diagram before?

Barbara: No, but it looks like part of the similar triangles in the double-angle formula I did before ... [referring to PWW-7]

I: What do you think it represents?

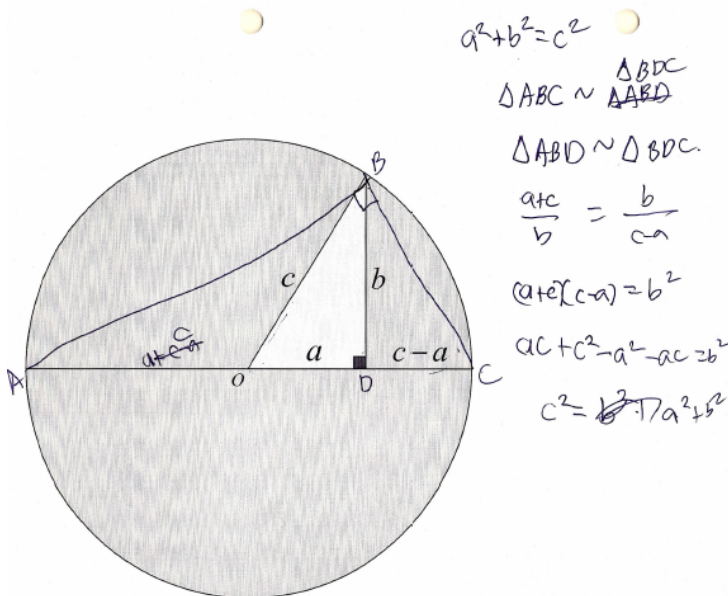
Barbara: Uhm ... Pythagorean Theorem?

I: What gave it away?

Barbara: The right triangle and the a, b, c .

I: Any thoughts on how to show it?

Barbara: You know, I can just make it look like that other problem I did ... like you know connect with lines and make triangles ... maybe like try similarity again.



Barbara connected the endpoints of the diameter with the vertex of the given triangle. She then stated the Pythagorean Theorem in symbolic form. She proceeded to identify triangles that she was going to examine for similarity. She examined each

pair and decided that it will be “easier” to work with $\triangle BAD$ and $\triangle BDC$ because “it does not look complicated” and then she proceeded to justify the angle congruence in order to establish similarity. Barbara mentioned that proving PWW-7 made PWW-10 easy.

When presented with PWW-7 Barbara did not recognize what the theorem was about.

- I: What do you think it [PWW-7] represents?
- Barbara: Something about trigonometry ... I do not remember much about trigonometry ...
- I: Try to describe what you see on the diagram and maybe it will come to you.
- Barbara: I see coordinates and the formula for ... oh, I see it's a unit circle, so the radius is 1. And I see this triangle [shading triangle] maybe I can do area 'cause it the area of this triangle is like uhm the base is $\cos 2\theta$ and the height is $\sin 2\theta$ Well, actually I don't know if that's what we're really looking for... what am I looking for?
- I: Here is a hint ... find $\sin 2\theta$ and $\cos 2\theta$ do you remember any of the trigonometric identities?

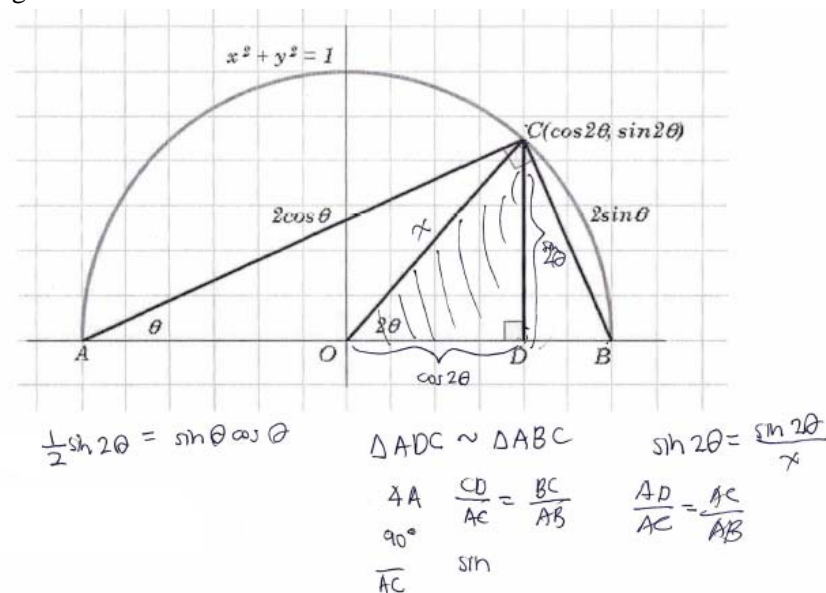


Figure 15 – Barbara's work on PWW-7

Barbara realized that $\triangle ABC$ and $\triangle ACD$ share $\angle \theta$ and both of them are right-angle triangles, therefore they are similar. However, Barbara confused the requirements for similarity and congruence. She mentioned that since these triangles have congruent angles and a common side they are similar, but in fact she was stating the reason for congruence not similarity. She then proceeded to establish corresponding sides, ratios and finally she expressed $\sin 2\theta$. She did not continue to find $\cos 2\theta$ she seemed to be overwhelmed by

the complexity. Barbara admitted that she never saw any derivations of trigonometric functions that she just memorized them.

Barbara was more successful with visual theorems illustrating algebraic and number theoretical results. She said about the algebra content that the diagrams are showing algebra tiles and that it is very easy. She remembered formulas for the sum of n consecutive integers and the sum of cubes; however, even though she was able to write numeric expressions explaining the geometric sums Barbara could not write them down in symbolic form, and she could not prove it algebraically.

PETER came to the United States to obtain a graduate degree after completing an undergraduate program majoring in mathematics and science in his native country. Peter was not interested in teaching secondary mathematics; he planned to obtain a doctorate degree and become an academic. Although Peter's mastery of English was not strong it was sufficient to study at the graduate level. Peter spoke several dialects of his native tongue and he also spoke French.

Peter reported that his high school mathematics program had allocated a lot of time to algebra, geometry, trigonometry, and calculus, but no other subjects. He indicated that he learned "ruler and compass" constructions, and did a lot of geometric proofs in high school; however, "... they were all computational and about practical things, like best angles for the bridge ...". In his undergraduate program, in addition to the Calculus sequence, Peter took courses in geometry and linear algebra, analysis, and number theory. In addition to practicum teaching he gained some teaching experience tutoring his peers and high school

students. Peter took one mathematics methods course. His GS = 6, PSVT = 13(54%), and he attained vHL1.

Since this study involved pre-service mathematics teachers rather than high school students, the investigator chose to accept the stricter criterion (SC) for van Hiele levels attainment and therefore, since Peter scored (5, 3, 3, 1, 2) respectively on the van Hiele Geometry Test his attainment level was vHL1. He generally had difficulties with distinguishing between a proposition and its converse, and necessary and sufficient conditions.

During the clinical interview Peter was able to recognize all visual theorems. He was able to write down the symbolic statement for each of the geometry, algebra, and trigonometry theorems, but he was able to write down only numeric statements for the geometric sums and the number theoretical problems.

When asked to describe the diagram for PWW-1 and show how he would prove it Peter wrote word “Area” and then $c^2 = a^2 + b^2$

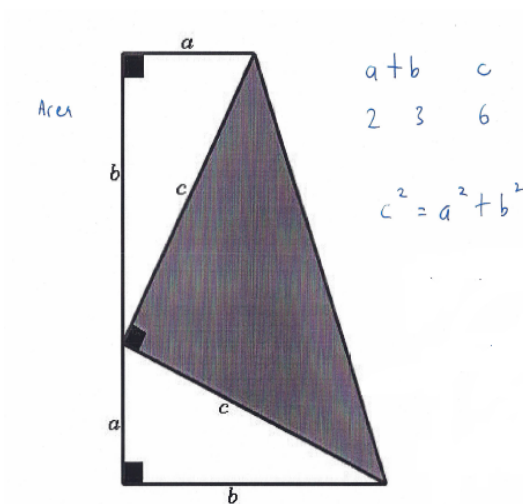
I: From what is given to you on the diagram do you think you will be able to prove it... or to explain it?

Peter: Oh okay. ... I need to prove the formula?

I: Yes

Peter: If I were to try to take the $\angle a + \angle c$ and try to see the connection with the hypotenuse length. So $a + b$ and c . So, if I were I see that a and b measurements I will try to measure the diagram. ... Just the side of a and the side of b and then measure the side of c

He then assigned the values of 2 to a , 3 to b and 6 to c . When asked about it, Peter



insisted that these might be the measurements of the sides and that he would physically have to measure to know definitely.

When the investigator suggested that Peter should try and describe the diagram, the objects he sees and the relationships he might deduce from the diagrams Peter replied:

Peter: But then I can see that this is a trapezoid or so if I have sides a and b and then a or b more than b . So I try to reason that the side of a and b will form...I will know that the sides of this a and b they can be parallel from what I see ...

I: How do you know they are parallel?

Peter: Because I am trying to use some angles that I know ...

I: Which angles are you using? Can you name them? Can you give the reason why you are using these angles?

Peter: Because I am trying to use a connection ... maybe I can do something with this trapezoid ... I just trying to find a connection. Uhm, I should find it. I maybe have to re-cut b is like two parts of a so I break into 3 parts. From what I see ...uhm the length of b is about more than 2 times the length of a . So say I will try to re-cut b maybe $2a$ more than $2a$.

Peter kept returning to the idea of measurement and the length of sides of the given triangles even though he wrote “area” previously, he continued associating the Pythagorean Theorem with the length of sides. Peter did not really make the connection between the “side squared” and the “area of a square with a given side”.

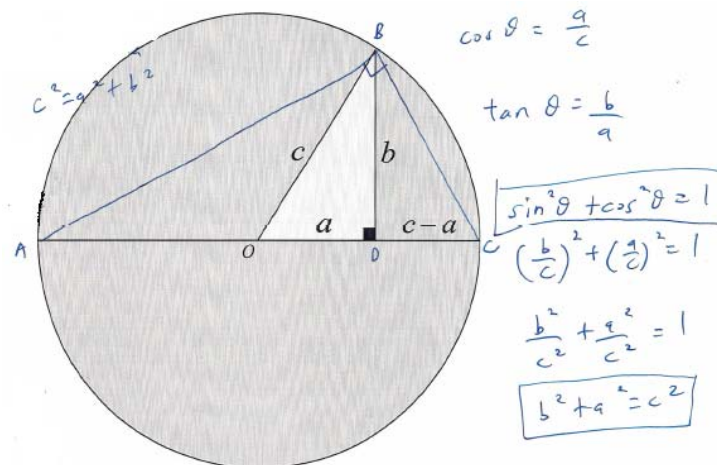


Figure 16 – Peter’s work on PWW-10

In PWW-10 Peter constructed a circular argument. He first defined trigonometric functions as ratios of sides a, b, c , then assumed the Pythagorean Identity, then substituted side ratios for the trigonometric function, performed an algebraic manipulation, and obtained the desired result. When asked how he would use the $(c-a)$ part of the diagram he replied that obviously it was not needed.

Peter successfully solved PWW-7: The Double-Angle Formula. He found $\sin 2\theta = 2\sin \theta \cos \theta$ and $\cos 2\theta = 2\cos^2 \theta - 1$ using similarity of triangles; however, he did not establish the similarity of any of the triangles in the diagram. He selected sides for finding ratios without a test, just by the look of it. When asked how he chose the sides for ratios, Peter replied that he rotated triangles in his mind until they were in the same position with respect to each other and that allowed him to observe corresponding sides.

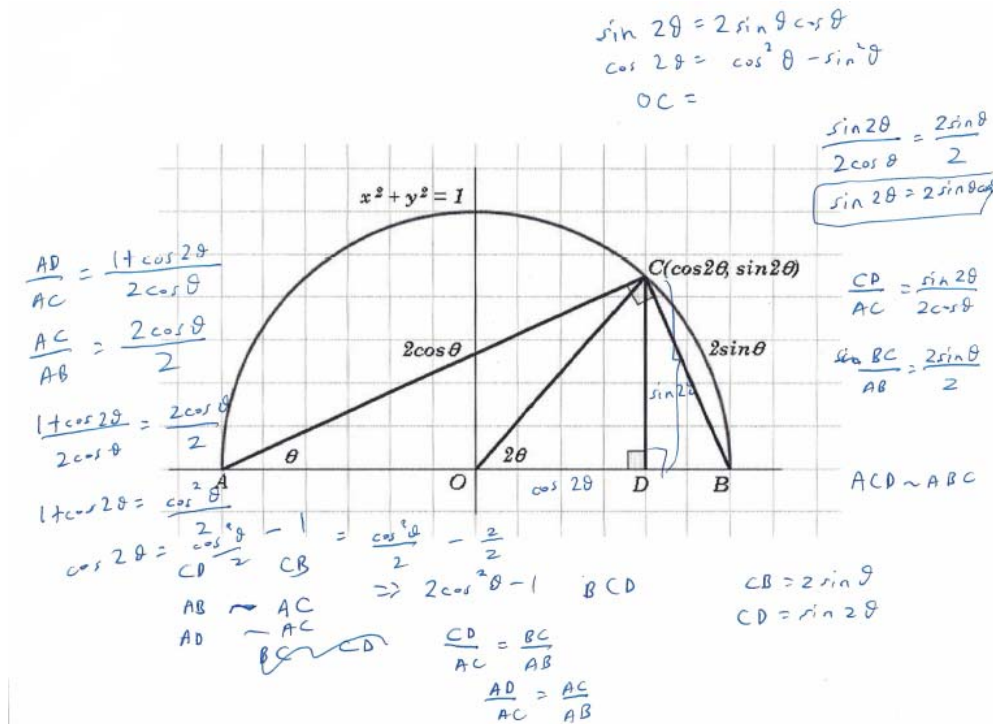


Figure 17 – Peter’s work on PWW-7

Peter was more comfortable with algebra and trigonometry problems. He was able to perform algebraic manipulations quickly and fluently because these required minimal explanation.

POLLY was educated in the United States. Although her first language was English she was fluent in Spanish because “... my family speaks Spanish and most of my friends speak Spanish”. She held a bachelor’s degree in education with a minor in mathematics. Polly had some teaching experience gained while tutoring in college in addition to Practicum experience in her teacher certification program. Upon graduation Polly intended to teach intermediate or senior mathematics. She also considered obtaining a doctorate and teaching at the college level. Polly reported that her high school mathematics program allocated a lot of time to studying algebra, functions and trigonometry, but only some time to studying

geometry; no time was allocated to other content areas. Polly reported that geometry was not taught as a separate course but was combined with trigonometry. She did not do “ruler and compass” constructions and when asked about it did not know what they were. Polly reported that in her high school mathematics program she never did any proofs and that the first time she had to prove anything was in her first semester of Calculus. Her favorite subjects were Algebra and Calculus, less favorite Trigonometry, and the least favorite was Geometry. Before taking mathematics core sequence in her undergraduate program Polly was required to take remedial mathematics and pre-calculus. Beyond high school Polly reported taking a geometry course, a linear algebra course, and a course in history of mathematics. Polly also took a problem-solving course and three methods courses. Her GS = 6, she attained vHL1 and she had the lowest PSVT = 12 (50%) score of the group.

Polly scored (5, 2, 3, 1, 2) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL1. She had difficulties with properties of geometric objects; she could not distinguish between a proposition and its converse, and she did not differentiate between necessary and sufficient conditions.

When presented visual theorems Polly was not able to recognize any of the problems presented. When asked whether she can guess what PWW-1 was about Polly said:

- I: Have you seen this diagram before?
- Polly: I might have. It looks familiar ... it is a figure with three triangles.
- I: What theorem do you think it represents?
- Polly: You mean like the actual name?
- I: If you do not recognize it perhaps, you can guess what it is about.

Polly: How a trapezoid can be formed using the properties of right triangles. You can take one triangle and rotate it and that makes another triangle.

When the investigator suggested that the diagram represents a proof of the Pythagorean Theorem, Polly wrote: $c^2 = a^2 + b^2$, but could not relate it to the diagram. Eventually, with investigator's help Polly established that she needs to find areas of the three triangles, but could not make the connection between the areas of the triangles and the area of the trapezoid.

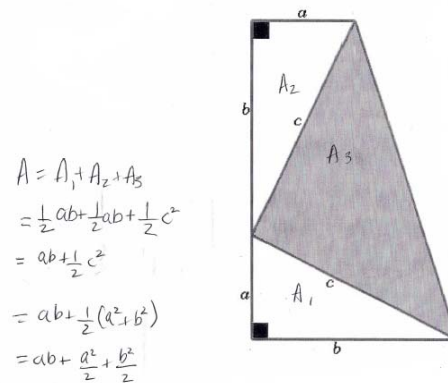


Figure 18 – Polly's work on PWW-1

When presented with PWW-10 Polly was able to guess that it is about the Pythagorean Theorem.

I: What visual theorem do you think this diagram represents?

Polly: Something with right triangle ... Pythagorean probably...

I: You guessed it. What made you think of the Pythagorean Theorem?

Polly: When I see a, b, c and a right triangle sign ... there's probably another method of proving Pythagorean Theorem using the circle.

However, she could not proceed with the solution and needed investigator's intervention.

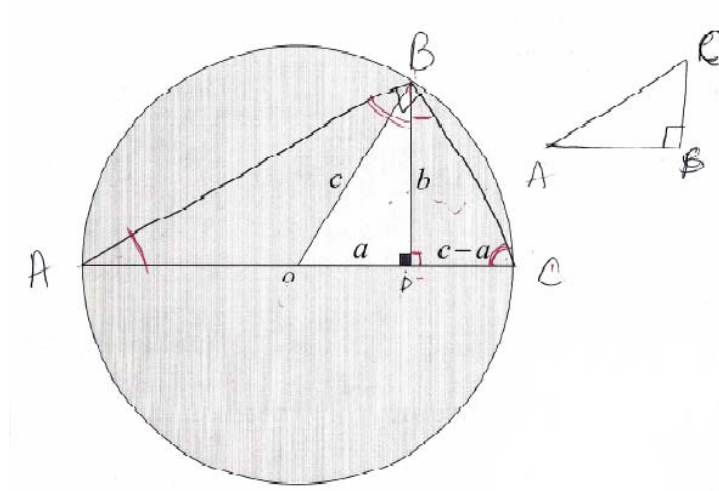


Figure 19 – Polly's work on PWW-10

- I: How are you going to proceed?
- Polly: [examines the diagram] Uhm, I wonder why it's $c - a$? I don't know. ... I don't know where to start ... can you give me a hint?
- I: [connects endpoints of the diameter with the vertex of the given triangle] Does this diagram look familiar?
- Polly: I am going to go with triangles I saw before [names vertices A, B, C, D marks $\angle B$ as a right angle] I am going to go with this big one, but I need to turn it around so it is like the other triangle

She did not define the problem space, did not articulate her goal, and did not devise a plan.

Moreover, when she was given help with the solution she could not show similarity of triangles. To determine corresponding sides, Polly re-drew one of the triangles in "standard position"; however, she did not name it properly and became more confused by the drawing. To justify their similarity she relied on the way they looked.

Polly: This big one and this little one are similar. [tracing $\triangle ABC$ and $\triangle OBD$]

I: How do you know they are similar?

Polly: AB is similar to OD and OB is similar to AC ...

I: But what tells you they are similar?

Polly: Because they are the same shape.

Polly had the same difficulty when solving one of the trigonometry problems, PWW-7: the Double-Angle Formula. In order to set up ratios she needed to identify corresponding sides of similar triangles; but since these triangles were not in “standard positions” Polly created sketches of these triangles, but did not label them properly because she did refer to corresponding angles to identify corresponding sides.

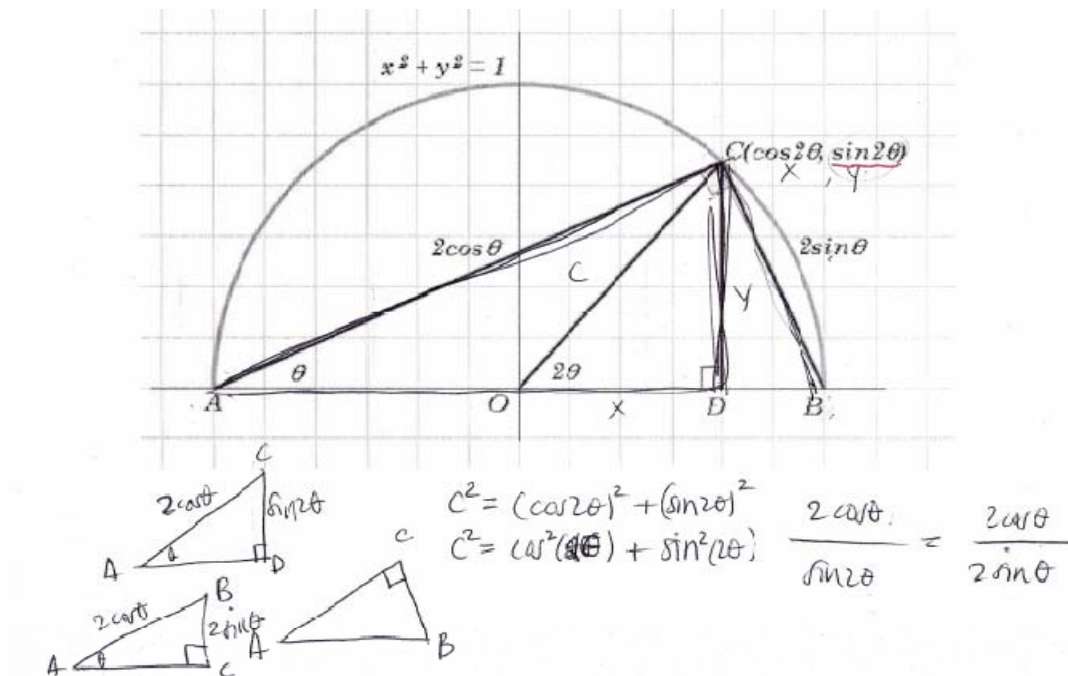


Figure 20 – Polly’s work on PWW-7

Polly was unsuccessful in solving other visual theorems. She was able to write a numeric interpretation of one of the geometric sums (PWW-9) when asked to describe what she

saw; however, she was unable to write a general statement using algebraic notation or give any kind of justifications using algebraic manipulation. Polly recognized the “square of the sum”, but did not recognize “completing the square” and when given help identifying it she could not complete algebraic proof of it.

MANDY immigrated to the United States several years ago. She was the oldest participant in the study. In her native country Mandy was an elementary school teacher. She held an equivalent of a bachelor’s degree in education with a minor in mathematics and an equivalent of a master’s degree in education. She had extensive teaching experience in elementary and middle school. Mandy had difficulties expressing herself in English; however, her reading and comprehension skills were strong enough to manage graduate level work in English. Mandy’s main goal was to obtain teacher certification in the United States, but she also wished “... to learn American culture and American methods of teaching children”. When asked about her undergraduate and graduate coursework she reported taking six years of mathematics and would not elaborate on specific content areas: “... in my country we learn everything a teacher should know to be a good teacher...” Mandy also took a problem-solving course and three methods courses. Mandy reported in her CV questionnaire that while in high school she studied geometry as a separated subject; she learned geometry with trigonometry; and that she studied “ruler and compass” constructions. Mandy’s $GS = 5$, she attained vHL1 and PSVT score was 15 (63%)

Mandy scored (5, 3, 1, 2, 2) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL1. She had difficulties with properties of geometric objects; she could not distinguish between a proposition and its converse, and she did not

differentiate between necessary and sufficient conditions. She was unsuccessful demonstrating understanding in interdependency of relations.

During her interview, Mandy would not engage in “thinking aloud” even though she understood that it was what she was asked to do; however, she would respond to questions when asked.

- I: When you look at the diagram, please, say what you see and what you think the meaning of it is. I would like you to think aloud when you solve these problems. ... I mean speak your thoughts ... Do the instructions make sense to you?
- Mandy: Yes, when I see geometry shape I have to say what it is and when I solve this problem I should tell you how I solve it...
- I: Have you seen this diagram before?
- Mandy: Yes.
- I: What theorem does it illustrate?
- Mandy: Theorem about triangles.
- I: Can you name it?
- Mandy: No.
- I: How do you know it is about triangles? [long pause]
- Mandy: Because I see triangles
- I: Can you describe what you see?
- Mandy: I see triangles
- I: Do you see other shapes?
- Mandy: Yes, I see a trapezoid.

She was able to identify geometric objects in the diagram and state the Pythagorean Theorem, but could not connect diagrammatic representations and algebraic representations.

- Mandy: I'm supposed to prove the Pythagorean theorem.
- I: Maybe if you state it first, maybe that will help.
- Mandy: Pythagorean theorem states that the sum of the squares of two sides is

equal to the square of the hypotenuse side.

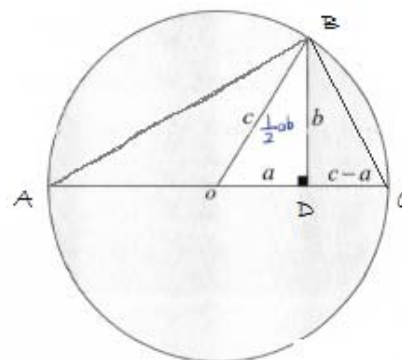
I: Right so if you write it down maybe...

Mandy: So, I have $a^2 + b^2 = c^2$ is what I am supposed to prove. [long pause]

I: ... and the Pythagorean theorem is generally about...? [long pause]

Mandy: ... A right triangle, right?

When presented with PWW-10, another diagram which illustrates the Pythagorean Theorem, Mandy did not recognize the diagram, did not connected information in the diagram to a previously established results, and when given help could not relate it to the diagram. She assumed



that this version of the Pythagorean Theorem is also about area and therefore, indicated on the diagram that the area of $\triangle OBD = \frac{1}{2}ab$. She ignored given $(c-a)$ and did not see the possibility of a different method for obtaining the desired result.

After the investigator suggested that additional construction may provide a hint in solving this problem, Mandy acknowledged that the diagram is similar to PWW-7 and attempted to set up ratios of sides. Here she encountered the same difficulties as in her attempt to solve PWW-7.

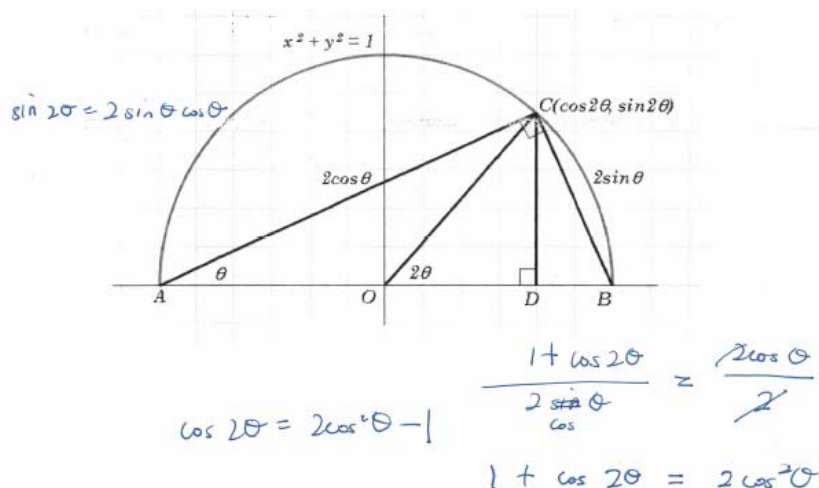


Figure 21 – Mandy's work on PW-7

While working on PW-7, Mandy, although unfamiliar with the diagram, was able to guess that it had to do with some trigonometric result. She was able to identify half of a unit circle, equation of the unit circle and the coordinates of point $C(\cos 2\theta, \sin 2\theta)$. When she was told that this visual theorem illustrates several trigonometric identities under the name “The Double-Angle Formula”, Mandy, wrote down $\sin 2\theta = 2 \sin \theta \cos \theta$, hence she knew the formula. However, when asked to explain how the diagram illustrates this result she had difficulties establishing the correspondence between, for example, lengths of $\sin 2\theta$ and the side CD of $\triangle OCD$.

I: $\sin 2\theta$ is a ratio of the opposite side of the triangle and its hypotenuse (in this case equal 1), can you identify these on the diagram?

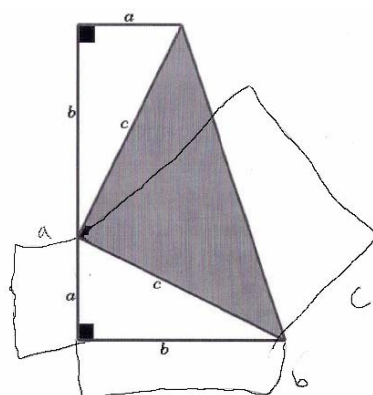
Mandy: ... When I look at the picture, I do not know what I am supposed to look for. Because I didn't see the measures of the angles and I didn't see the dimensions of the triangle so I do not have in mind what we are calculating or what we are looking for what formula we are looking for.

When solving other PW's, Mandy could not identify any of the visual theorems; however, when given the names she was able to state formulae or numeric expressions of

these theorems. She required help making connections between geometric objects and diagrammatic notation, geometric relationships and their algebraic expressions.

TIMMY was educated in the United States. His first and only language was English. Timmy held a bachelor's degree in mathematics with a minor in science and a master's degree in mathematics from a southern university. Upon graduation Timmy intended to teach secondary school mathematics while working on his doctorate. His eventual goal was to do research in mathematics education. Timmy had previously taught college mathematics and he had secondary practicum experience. Timmy reported that his high school mathematics program focused mainly on algebra and functions and relations. He did not study geometry in a separate course, did not do "ruler and compass" constructions and did not do any proofs while in high school. In his undergraduate program, in addition to the core mathematics sequence Timmy took four courses concerned with spatial relations. So far, in his teacher certification program he has completed one methods course and was enrolled in the second one; and he took a problem solving course. Timmy's GS = 6, PSVT = 19 (79%), and he attained vHL3.

Timmy scored (5, 4, 4, 1, 4) respectively on the van Hiele Geometry Test and since he did not attain vHL4 even though he scored high in vHL5, according to the fixed sequence property he was assigned vHL3. According to the analysis of behaviors at each van Hiele level (Usiskin, 1982, pp. 10-12) the lack of success in passing vHL4 signifies participant's inability to distinguish between a proposition and its converse; lack of understanding the difference between necessary and sufficient conditions; moreover, the lack of understanding of the need for axioms.



During the interview Timmy had difficulties thinking aloud. He had to be prompted to share his thoughts. Although he did not recognize PWW-1 from his past experience, Timmy quickly guessed that this visual theorem was about the Pythagorean Theorem.

I: What do you think this diagram represents?

Timmy: Probably the Pythagorean Theorem

I: Why do you think so?

Timmy: Right-angle triangles ...

I: Based on the information given in the diagram can you prove it?

Timmy: Right now I am just... if I am going to look at the areas cause I know a^2 gives me a rectangle, b^2 gives me a rectangle, c^2 gives me a rectangle... I was just drawing them. I was just kinda seeing how it would match up.

I: So you are trying to construct something ... a familiar proof of Pythagorean theorem.

Timmy: Yah I think so. I was just seeing how it matched up. So what do you want me to do with this picture?

Timmy was unsuccessful in developing a solution to PWW-1; he could not grasp the idea that a proof was required. When given a symbolic expression $\frac{1}{2}(a+b)^2 = ab + \frac{c^2}{2}$ Timmy thought it was obvious and required no proof.

When shown PWW-10, Timmy did not recognize it but guessed that it too might be a visual proof of the Pythagorean Theorem, again because of the right-angle triangle. However, he did not know how to proceed. When asked to describe the diagram Timmy decided to “cut up the circle” into parts to see if he could figure out the proof. When the

investigator offered Timmy a hint by connecting the endpoints of the diameter and the vertex of the triangle Timmy recognized right away that this problem is similar to PWW-7 and proceeded to identify similar triangles. However, he did not use the properties of the similarity, but rather re-drew $\triangle ABD$ and $\triangle BDC$ to determine their corresponding sides.

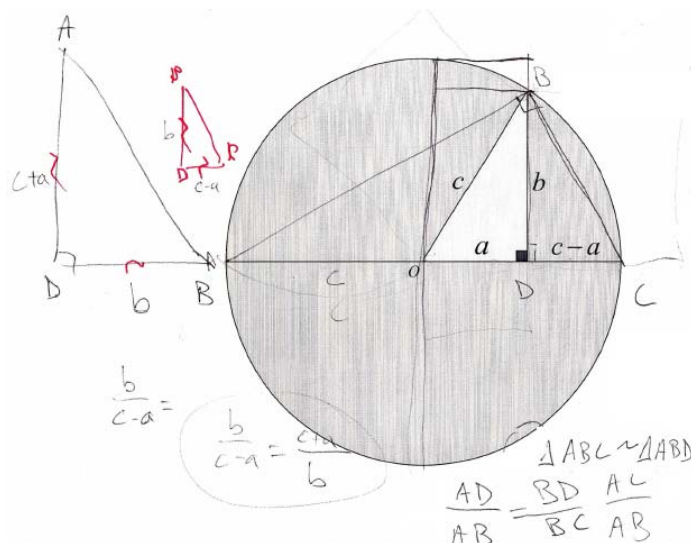


Figure 22 – Timmy's work on PWW-10

When working on PWW-7 Timmy was able to find the double-angle formula for $\sin 2\theta$; however, he was unable to set up ratios to express the $\cos 2\theta$. Since Timmy was not able to identify in similar triangles corresponding sides using the given the angles, he kept re-drawing triangles looking for a position that will give him insight. For example, he chose $\triangle ACD$ and $\triangle BCD$ and set up ratios $\frac{AD}{AC} = \frac{CD}{BC}$, but $AD = 1 + \cos 2\theta \neq \cos 2\theta$. Then he decided to use $\triangle ABC$ and $\triangle ACD$, but again without corresponding angles he could not find corresponding sides.

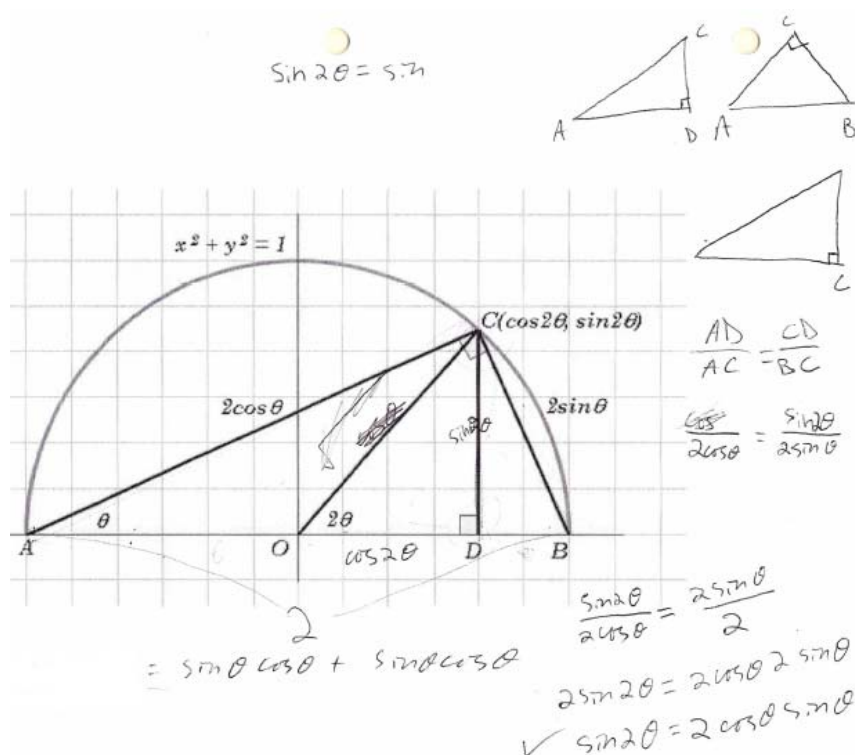


Figure 23 – Timmy’s work on PWW-7

Timmy was able to explain the “square of the sum”, “completing the square”, and the number theory problems; however, he could not write geometric series problems in symbolic form and could not prove it.

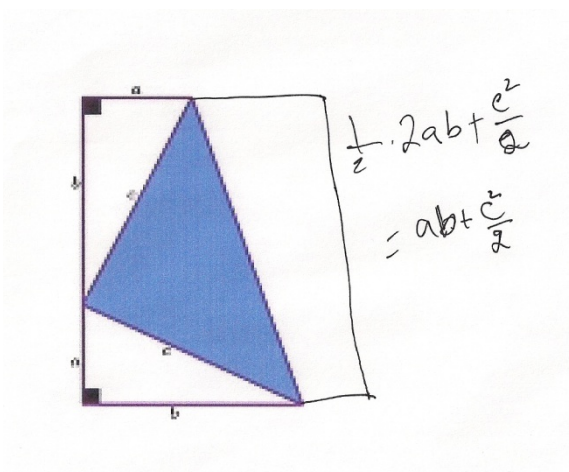
KATIE was educated in the United States. English was her first and only language. Katie held a bachelor’s degree in mathematics with a minor in political science and a professional degree from a prominent university. She returned back to school to earn her master’s degree in mathematics education and to obtain teacher certification. Upon graduation she intended to teach secondary mathematics; she said that her professional degree was a detour on the way to her true calling – being a mathematics teacher. Katie had some experience teaching mathematics, which she gained while tutoring high school students to supplement her income. In high school Katie took AP Calculus. She reported that her

mathematics program allocated a lot of time to algebra, geometry, functions and relations, and calculus. The program also allocated a fair amount of time to probability and statistics. Although her high school geometry was taught as a separate course she did not learn geometry with trigonometry; and she did not study “ruler and compass” constructions in middle school or in high school. She did not remember doing any proofs in high school. In her undergraduate program in addition to core courses, Katie took one semester of topology. In her graduate program Katie took a problem-solving course which she enjoyed very much. Katie’s GS = 5, PSVT = 21(88%), and she attained vHL3.

Katie scored (5, 4, 4, 3, 4) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL3. Although she attained vHL5 based on the weaker criterion since she is a pre-service mathematics teacher she does not meet attainment level based on the stricter criterion for vHL4 and therefore is assigned vHL3.

- I: Have you seen this diagram before?
- Katie: No, ... I don't think so
- I: See if you could guess what it is about
- Katie: [looks at the diagram in silence]
- I: Perhaps, if you share with me your thinking it might come to you
- Katie: I cannot guess ... congruent triangles?
- I: Can you describe what you see?
- Katie: I see triangles ... one is blue and two are not ...
- I: Anything else?
- Katie: I see two right angles and I see this figure ... [traces the trapezoid]
- I: Does this figure have a name?
- Katie: Oh yeah ... It's a trapezoid ... [thinking in silence]
- I: Why don't you share with me what you are thinking of and perhaps an idea will come to you as you say it ... as if I was a student and you were a teacher. Is there a relationship between these triangles and this trapezoid?
- Katie: [thinking in silence]

When told that the diagram is about the Pythagorean Theorem, Katie intuitively tried to construct the other half of the circumscribed square; however, she did not remember to complete the inscribed square and had to abandon the idea. She then proceeded to express the areas of the



triangles in symbolic form; therefore, even though her thinking was disjointed she did recognize the connection between AREA and the Pythagorean Theorem. However, she did not see the need to prove that the isosceles triangle in the diagram was a right triangle since she just wrote a symbolic expression for the area of this triangle without any justification. She was able to proceed with the solution when told that the three triangles tile the trapezoid and if she could find the areas she will be able to solve it. Her solution consisted of algebraic manipulations of symbolic statements.

I: Can you describe what you are trying to do?

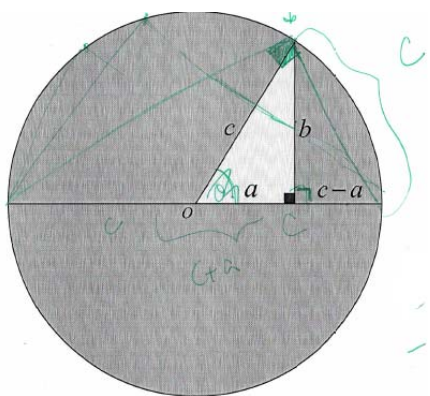
Katie: Uhm, I'm looking at the three triangles and I'm trying to figure out how ... I just don't understand I haven't obviously I haven't seen this before so I don't... I'm thinking the Pythagorean Theorem so... So I'm starting with $a^2 + b^2$...[pauses] ... Now, I'm starting to get confused about the Pythagorean Theorem [Laughs]. Wait uhm $a^2 + b^2 = c^2$

I: Let me show you a symbolic statement that represents a proof of this visual theorem and you will try and explain how it came about...

$$ab + \frac{c^2}{2} = \frac{(a+b)^2}{2}$$

Katie: Okay, so I'm reading the equation that you said it represents the diagram that proves it. ... It takes uhm the two triangles based on the fact that they're uhm $a \times b$ would be in the rectangle. So half of that would be the area of each white triangle plus the half of the c^2 . So then it would be the uhm half of the big square and then it just factors

out the $\frac{1}{2}(a+b)^2$ and then therefore c^2 equals ... I'm missing this part. I have to do the math in my head. I have to write it down
[Laughs]



Katie was not successful solving PWW-10, the other visual proof of the Pythagorean Theorem, even though she was able to do additional construction. When asked to describe what she saw on the diagram Katie stated that she saw a circle and a triangle with sides a, b, c and if it was not for

the $(c-a)$ she would think it was about the Pythagorean Theorem, but that other notation confused her. She could not see the similarity between PWW-10 and PWW-7. This problem was left unfinished.

When working on PWW-7 Katie was able to use similarity of triangles to find ratios of sides and express both $\sin 2\theta$ and $\cos 2\theta$ in terms of other values given in the diagram. However, she assumed similarity, she did not offer any justification for the similarity of the triangles, and she established the correspondence of sides through mental rotations.

Katie had no difficulties explaining the “square of the sum” and “completing the square” problems she was able to write symbolic statements for the sum of n consecutive integers and the sum of cubes; however, she could not explain the geometric sums problems, and did not even translate the image into a numeric statement.

ABIGAIL was born and educated in the United States. Her first language was English. She did not speak any other languages. Her undergraduate major was English she had no minor concentration. During her interview Abigail stated that she always liked mathematics but was never comfortable enough to pursue a degree in mathematics because she was “afraid of proofs”. Upon graduation she intended to teach at the elementary or intermediate level. Abigail considered a doctorate in education with the intention to teach elementary school teachers. Her teaching experience was limited to teaching practicum. Abigail reported that her least favorite subject of study was Geometry and her most favorite was Calculus. She reported that her high school mathematics program allocated a lot of time to studying algebra and functions, and some time studying probability and statistics. Abigail reported studying geometry in high school as a separate course, but she does not remember studying geometry with trigonometry and she does not remember studying “ruler and compass” constructions. When asked about “ruler and compass” constructions she replied: “How can you construct anything with a compass, I thought it is supposed to help you orient yourself when you are like in the mountains or in a forest ...”. Abigail reported that in high school “... we never did any proofs, so when I got to college it was really difficult”. Abigail started her undergraduate study of mathematics in remediation and was required to take a pre-calculus course. She did not take any geometry or linear algebra courses or any other courses that study spatial relations such as topology or graph theory in her undergraduate program. She said she was only required to take differential and integral calculus sequence. While in graduate school she took a problem solving course that had a lot of Geometry content and while she was struggling in the course she found the material

very interesting and wished she was better prepared for it. Abigail reported taking two methods courses. She scored 79% on PSVT, acquired vHL3, and was assigned $GS = 6$.

Abigail scored (5, 5, 4, 1, 5) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL3. Although she attained vHL5 because of the fixed sequence property of the van Hiele model, since she did not attain vHL4, she was assigned vHL3. Based on the test-item analysis Abigail does not understand the difference between a proposition and its converse, and she does not distinguish between necessary and sufficient conditions.

During the clinical interviews Abigail consistently misidentified geometric objects, for example, when working on PWW-1 she called the trapezoid a rectangle and when asked why she thinks it is a rectangle she corrected herself; or she identified a triangle with two sides of length c and unknown third side – an equilateral. When attempting to compute the area of the triangles, she identified the lengths of the sides given on the diagram and asked if she could assume that the sides are equal in measure.

I: Can you say what you see?

Abigail: Okay. Uhm... so, I guess this would be in generic terms oh wait so is this triangle [tracing one of the right triangles] congruent to this triangle [tracing the other right triangle] since this is a , b , c [pointing to annotated sides] and this is a , b , c ? Can I assume that?

I: Please, continue ...

Abigail: Okay, okay. Oh, and this must be equilateral because this is c and this is c ? I suppose.

She did not see the need to show that the isosceles triangle with two congruent sides measuring c is a right triangle. After expressing the sum of the areas of the triangles in

symbolic form and simplifying the expression she wrote on the side “Q.E.D.”. When it was suggested by the investigator that the solution is incomplete and that now the area of a trapezoid needs to be found, Abigail decided to divide up the trapezoid into a rectangle whose area she thought she was going to be able to compute, but then she abandoned this route because she did not know how to compute the area of the resulting triangle.

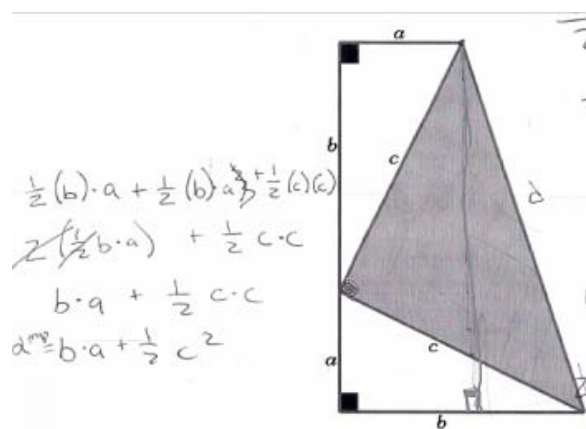


Figure 24 – Abigail’s work on PWW-1

Abigail: There I could do the area of this rectangle ...uhm ... plus the area of this triangle. But how I get the area of this triangle? [long pause]

I: Perhaps you could compute the area of the trapezoid

Abigail: Trapezoid? I don’t know ... ‘Cause I don’t know the formula for the area of a trapezoid [Laughs].

I: Would you like me to tell you the formula?

Abigail: Okay

I: The average of the parallel sides multiplied by the altitude of the trapezoid

Abigail: Oh average of these two. Okay average of parallel sides. I have no idea how to write that in mathematical notation.

When working on the visual proof PWW-7 Double-Angle Formula, Abigail could not write down the ratios because she could not name corresponding sides of similar triangles (they were not in standard position). Moreover, even though Abigail could interpret geometric notation such as filled-in square at the base of the right angle, by her own admission she

would not know how to tell whether the angle in question was a right angle if the right angle was not indicated on the diagram.

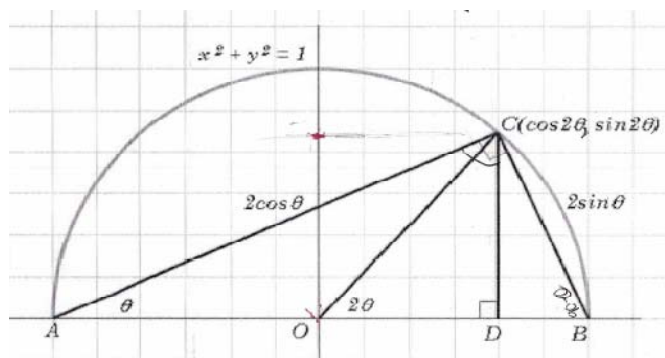


Figure 25 – Abigail's work on PWW-7

Abigail: Okay, well $\angle ACB$ is a right angle

I: Yes, how do you know?

Abigail: Because it has a little box inside

I: But if it did not have a little box would you know?

Abigail: I wouldn't ... would you?

I: In this case yes I would because this angle subtends the diameter of the circle

Abigail: What did you say? What's that word mean?

When working on geometric sums, Abigail was able to recognize that this visual theorem demonstrates the sum of the shaded squares yet she could not write it as a formal statement and she could not state the limit of this geometric series.

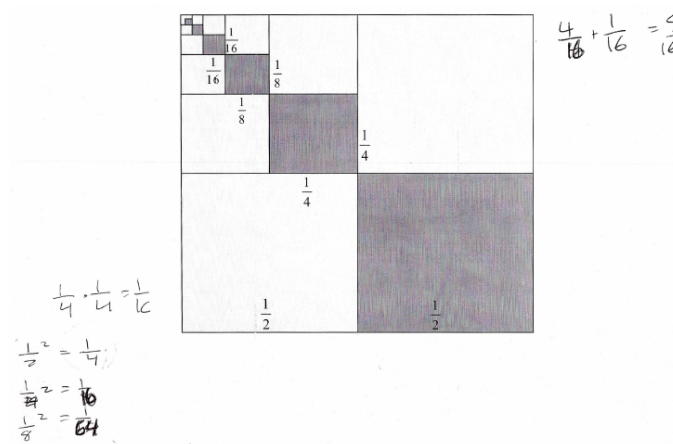


Figure 26 – Abigail’s work on PWW-5

JENNY was educated in the United States. Although she reported her first language was English she spoke a mixture of English and Spanish with her parents and her friends. Jenny held a bachelor's degree in Spanish with a minor in mathematics. Upon graduation from the teacher certification program she intended to teach intermediate or senior mathematics. In high school Jenny took what is called integrated mathematics. A sequence of courses focused mainly on algebraic skills with a mix of functions and relations; a study of geometry was devoted very little time. Jenny did not study “ruler and compass” constructions; she reported that she did not do proofs in her high school mathematics courses: “... We were not required to do proofs”. Jenny started her undergraduate mathematics program in remediation followed by a course in pre-calculus before she could begin mathematics core sequence. Jenny reported taking a linear algebra course in her undergraduate program but no geometry, or topology, or graph theory. In her graduate program Jenny’s only encounter with geometry was through a problem solving course which “... is so hard, I am really worried about the grade ... I just need someone to tutor me ...”. When asked if she will be comfortable teaching a geometry course in middle

school or high school Jenny replied: "... I think I can manage, I will learn if I have to, I will plan and prepare ...". Jenny's GS = 4, PSVT = 19 (79%), and she attained vHL3.

Jenny scored (5, 5, 5, 1, 3) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL3. Based on the test-item analysis Abigail does not understand the difference between a proposition and its converse, and she does not distinguish between necessary and sufficient conditions. She does not demonstrate understanding of the role of axioms in mathematical deduction.

During clinical interview Jenny could not recognize or guess any of the visual theorems presented. She had difficulties describing the diagrams.

I: What do you think this is about?

Jenny: $a^2 + b^2 = c^2$

I: What do you see in the diagram that tells you that?

Jenny: These three triangles

I: Do you know the name of this theorem?

Jenny: ... Pythagorean Theorem?

I: Given the information in the diagram can you prove this theorem?

Jenny: Uhm, yeah. I think you have to demonstrate that the rotation of either of the purple triangles uhm would ultimately I think become the blue triangle. So like show that the area is the same.

I: Do you think the areas of these two triangles with sides a, b, c are equal to the area of the blue triangle with two sides c ?

Jenny: I don't know, maybe. I don't know if they fit this triangle maybe they bisect it.

... I know. I know that you would be able to do it. I'm just not entirely sure how because these are your a^2 and b^2 so they're equal in this case which is tricky. Has to be equal to this one the long side which I don't know how to make that equal.

When given a symbolic statement that represented the relationship in the diagram Jenny was not able to relate it to the diagram. Jenny explained that she does not know the formula for the area of the trapezoid and could not understand how these were equal.

I: Do you remember the formula for the area of the trapezoid?

Jenny: Uhm the big base times the little base is it? Times $\frac{1}{2}$ that could be area. Okay. So that's the area of a trapezoid, okay. [investigator wrote down the formula] So the big base oh so that's what that is the big base plus the little base squared times $\frac{1}{2}$ is area of a trapezoid ... I did not know that ...

She was not able to recognize PWW-10, although she guessed correctly that it was about the Pythagorean Theorem. When the investigator offered a hint by doing additional construction Jenny did not recognize the similarity of PWW-10 and PWW-7, the only visual theorem Jenny was able to work out.

I: Have you seen this before? [PWW-7]

Jenny: Yeah, and I can't prove it.

I: You don't need to prove anything right now you just need to tell me what you see

Jenny: Uhm, this is well a series of triangles that are constructed within a semi-circle ... You have actually constructed in this case a right triangle which uhm supposedly should be able to any time you construct a triangle with the uhm N points at the ends of the semi-circle. ... Uhm and the third angle somewhere on the semi-circle it should be a right triangle

I: Why?

Jenny: I don't know but I know that it's true because I had it on a test one time and I couldn't prove it

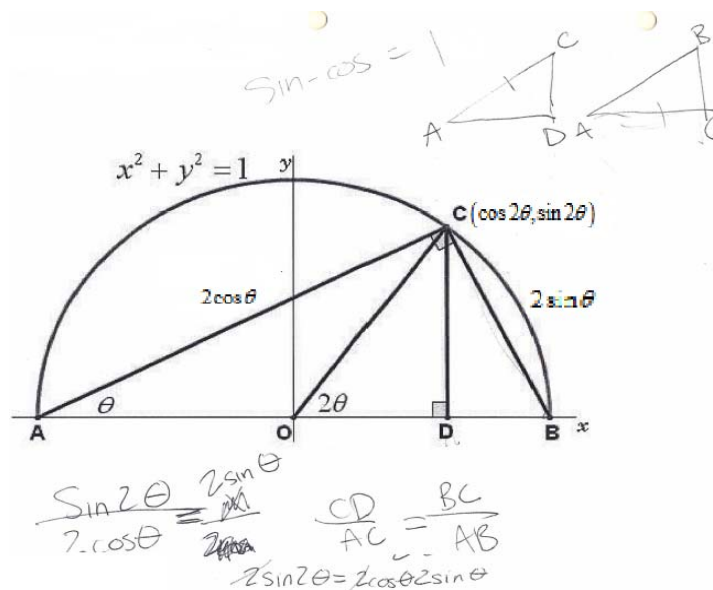


Figure 27 – Jenny’s work on PWW-7

Jenny could not justify that $\triangle ABC$ was a right-angle triangle. With some help from the investigator she realized that this visual theorem may be proven by using triangle similarity; however she could not determine corresponding sides and could not set up ratios to find $\cos 2\theta$. In order to find corresponding sides she drew triangles on the side in “standard position”.

Jenny had difficulties with the rest of the problems. She did not recognize the “square of the sum”, she said it was about “algebra tiles that are so popular now”; and attributed “completing the square” to the algebra tiles as well. She was not able to recognize the sum of n consecutive integers. When asked to describe the diagram she thought it was a game of “connect the dots” with a “jugged-edge line” going through it like a staircase.

ASHLEY was educated in the United States. She spoke no other languages, but English.

Ashley held a bachelor’s degree in mathematics with a minor in computer science. Upon

graduation, Ashley intended to teach intermediate or secondary mathematics. She had no prior teaching experience other than the practicum teaching. Her high school mathematics program allocated a lot of time to algebra and fair amount of time to geometry with trigonometry, i.e., geometry was taught as a separate course. Very little time was allocated to probability and statistics or functions and relations. Ashley did not do “ruler and compass” constructions and did not do any proofs while in high school. Ashley started her undergraduate program in remediation and was required to take a pre-calculus course. She took one courses in geometry in addition to the core mathematics sequence “... because my college required it for teacher education”; she did not take any other courses dedicated to the study of spatial relations other than what was part of the mathematics major core. So far, in her teacher certification program she has completed two mathematics methods courses but did not take any problem solving courses. Ashley’s GS = 7 and her PSVT = 20 (83%)

Ashley scored (5, 5, 5, 1, 4) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL3. Based on the test-item analysis Ashley does not understand the difference between a proposition and its converse, and she does not distinguish between necessary and sufficient conditions. She does not demonstrate understanding of the role of axioms in mathematical deduction.

During the clinical interview Ashley was asked to think aloud as worked on proving visual theorems. She was shown PWW-1 slide and asked if she recognized it. She did. Ashley shared that she recently taught the Pythagorean Theorem in her Practicum and even though the diagram was different it was also about the Pythagorean Theorem.

I: So then you can prove it?

Ashley: Not exactly with this image but uhm we kind of hinted at the normal where we have one triangle and then we go to the other three sides and have the boxes use them like grid paper.¹⁷ ... Actually I do not think so

I: Can you describe what you see in the diagram?

Ashley: Okay, I see 3 right triangles. Triangle 1, 2, and the blue triangle being last. And then I see this is a, b, c and then a, b, c . So I'm figuring this is this is saying that $c = c$ here and... So this would be more like uhm saying that our $a = b$ which is represented as our c . So then the hypotenuse would be like a c' . And so this would be our a^2 and then this would be b^2 and then this is our c^2 . So it's giving us the length. We're adding $a + b = c$ and then we have it again in triangle

Ashley continued deciphering the diagram until the investigator suggested that she looks at it from a general stand point. Then Ashley realized that she sees two objects a trapezoid and a group of triangles. Once she realized it, Ashley was able to represent areas and solve the problem; however, she did assume that the isosceles triangle is a right-angled triangle and offered no justification regarding it.

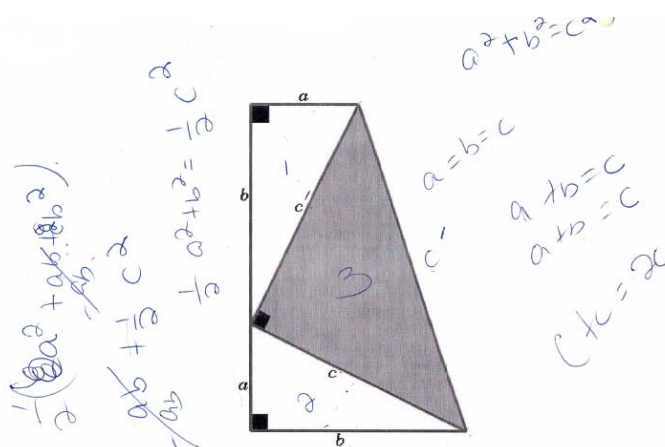


Figure 28 – Ashley's work on PWW-1

¹⁷ Ashley was describing a standard proof of the Pythagorean Theorem done usually at the end of elementary or beginning of middle school. It is done by cutting up squares constructed on the sides of a triangle and tiling the square on the hypotenuse of this triangle.

When shown PWW-10 slide, Ashley did not recognize the Pythagorean Theorem, but she did recognize that she has seen something similar a few slides back and decided to use the same technique, i.e., similarity of triangles to solve it. She performed additional construction, identified similar triangles, but offered no justification for their similarity. She could not set up the ratios by herself, but with the help of the investigator she was able to find a solution.

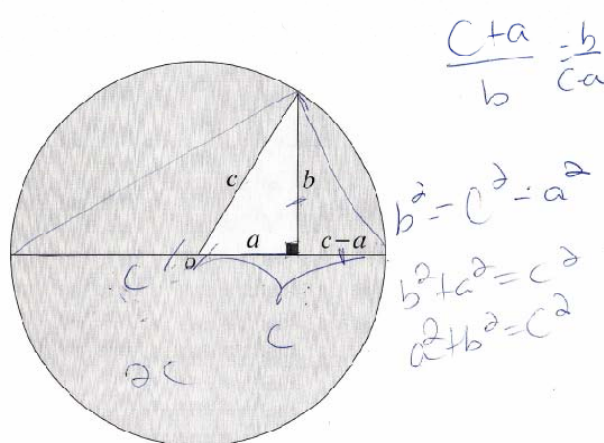


Figure 29 – Ashley's work on PWW-10

When working on PWW-7 Ashley had difficulties identifying similar triangles. She offered no justification for their similarity, and therefore, she had difficulties setting up ratios of sides. Ashley said she did not remember the trigonometric identities and did not know what she was working towards. She gave up eventually. Ashley complained that she could not visualize triangles in proper position to be able to identify corresponding sides.

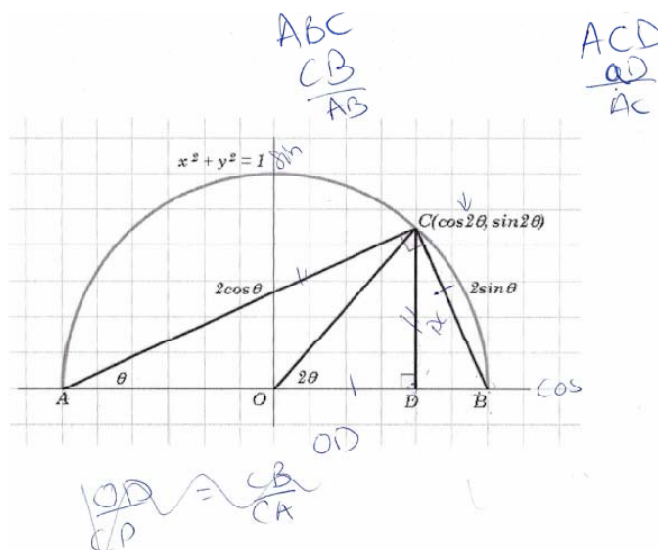


Figure 30 – Ashley’s work on PWW-7

When working on the remaining visual theorems Ashley was able to identify the theorems, but did not state them in symbolic notation. She felt satisfied by showing that numeric relationships held.

NANCY was educated in the United States. Her first language was English and her second language was French. She held a bachelor's degree in mathematics with no minor concentration. She had limited teaching experience; at the time of the study she has only completed her elementary school Practicum. Upon graduation with a master's degree in Mathematics Education Nancy intended to teach intermediate or senior mathematics. Nancy reported that her high school mathematics program allocated a lot of time for algebra, geometry, functions and relations; and fair amount of time to probability and statistics, calculus, and even linear algebra. Geometry was taught as a separate subject. Although she did not remember doing “ruler and compass” constructions in high school, she did remember these constructions from middle school curriculum. Moreover, Nancy reported “...doing a lot geometric proofs, like two-column proofs...” while in high school.

Nancy did not take an undergraduate level geometry course; however she did take four semesters of other spatial relations courses. In her graduate program Nancy took a problem-solving course and a mathematics methods course. She said that even though she is a “self-proclaimed algebraist” she enjoyed the problem-solving course because it had a lot of geometry content. Nancy’s GS = 8 and her PSVT = 17 (71%)

Nancy scored (5, 5, 5, 3, 5) respectively on the van Hiele Geometry Test and therefore, her level attained was assigned vHL3. However, it is worth mentioning that if the levels were assigned based on the weak criterion (WC), Nancy would have attained vHL5. Based on the test-item analysis Nancy had difficulty distinguishing between a proposition and its converse. She did not demonstrate understanding of the role of axioms in mathematical deduction.

During the interview Nancy was able to recognize most of the visual theorems, but she struggled to solve them. She was able to describe the PWW-1 diagram and consequently was able to find a solution. However, she did not offer a justification for computing the area of the isosceles triangle as $\frac{c^2}{2}$. Nancy assumed it was a right-angled triangle.

Although she developed a sketch of the solution she did not have a plan and she did not state her goal.

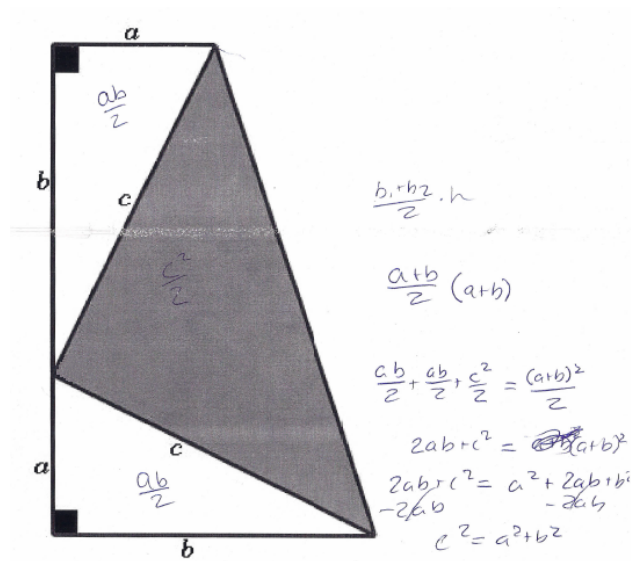


Figure 31 – Nancy's work on PWW-1

When solving PWW-10 Nancy attempted to do additional construction, but she was not able to see it through because her construction introduced more unknowns. Because she was focused on her construction she did not recognize that she already saw a similar construction PWW-7 and could have reproduced the diagram. Then she assumed what she had to prove. When neither approach worked she abandoned the problem.

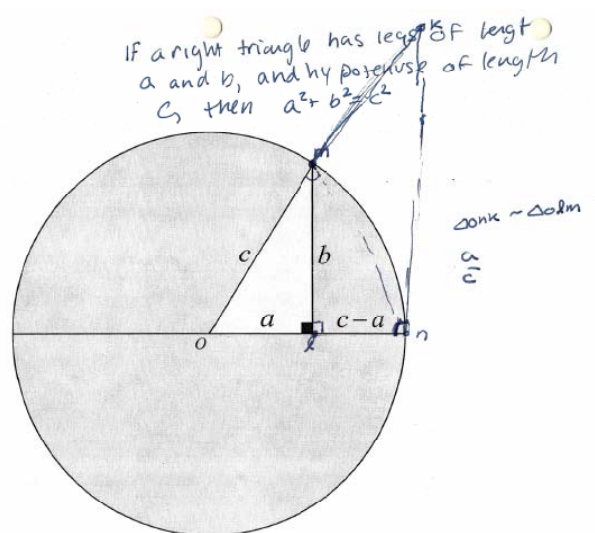


Figure 32 – Nancy's work on PWW-10

Although she was able to guess that PWW-7 was about finding trigonometric identities, when describing the diagram Nancy kept focusing on irrelevant information, such as trying to remember trigonometric identities instead of reading the diagram. She examined two triangles $\triangle ACD$ and $\triangle OCD$ in both of them CD is opposite $\angle \theta$ and $\angle 2\theta$ at the same time. Nancy could not resolve this seeming contradiction and continued working on it until she abandoned this visual proof.

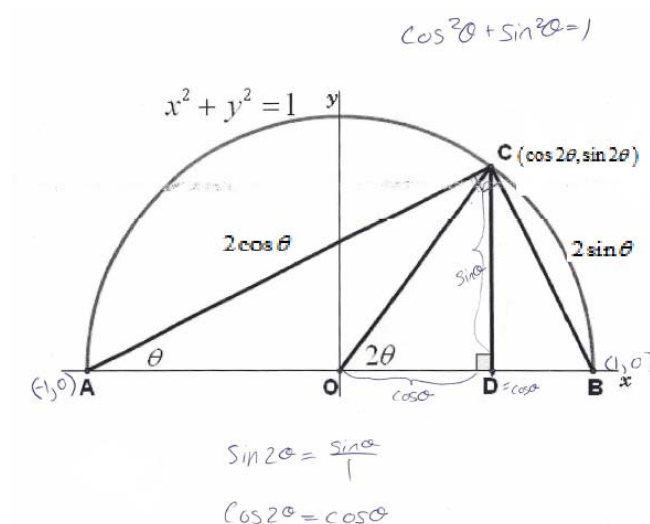


Figure 33 – Nancy's work on PWW-7

However, the rest of the problems she was able to represent in symbolic notation although she did not offer any type of proof for the geometric sums and for the sum of cubes she did mention that the latter should be proved by method of mathematical induction.

JERRY was born and educated overseas. He came to the United States to study in the graduate program. Although he spoke with a thick accent his mastery of English grammar was impressive. He also reported speaking two other languages. Jerry wanted to become a high school mathematics teacher so that he could work in the United States, learn how

American educational system functions, gain experience working with adolescents all while studying for his doctorate. Upon receiving his doctorate Jerry intended to go back to his native country and teach teachers. Moreover, he had administrative even political ambitions. Jerry reported that his high school mathematics program had allocated a lot of time to algebra, geometry, trigonometry, and calculus; a fair amount of time to probability, finite mathematics, and linear algebra. Jerry, according to his responses, studied all subjects included in the questionnaire at least a fair amount of time. He indicated that he learned “ruler and compass” constructions, and did a lot of proofs in all his courses. Jerry reported taking undergraduate courses in every content area indicated on the questionnaire, 24 mathematics courses in total in addition to 4 methods courses. When he found out that this study was concerned with geometric knowledge he volunteered: “... I like Geometry most of all ... I am very good in Geometry”. When asked whether he found studying at a major American university difficult, Jerry responded: “... not difficult, very easy ... I know all this already ... “. His $GS = 9$, $PSVT = 21(88\%)$, and he attained vHL3.

Although Jerry answered all questions in vHL1, vHL2, vHL3 and vHL5, he answered only three questions correctly in vHL4. Jerry could not distinguish between a proposition and its converse, and he had difficulties with the interdependence of relations (further addressed in Chapter 5 – Discussion).

When presented with visual theorems, Jerry did not recognize any of them, but he was able to describe the diagrams and through descriptions come up with solutions. He guessed that PWW-1 illustrated the Pythagorean Theorem.

- I: Can you guess what theorem it illustrates?
- Jerry: Pythagorean Theorem.
- I: Why do you think so?
- Jerry: I see right triangles, I see sides a, b, c ... I think maybe Pythagorean Theorem... so I will try.

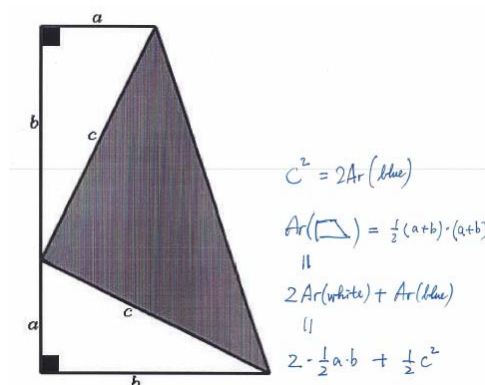


Figure 34 – Jerry's work on PWW-1

Jerry created a sketch of the solution. He observed that the area of the trapezoid is equal to the sum of the areas of the triangles. Although he assumed that the triangle with side c is a right triangle and offered no justification for it. Moreover, Jerry did not offer a more formalized solution; he did not work out all necessary algebraic manipulations that showed how the area of the trapezoid is equal to the sum of the areas of the triangles.

Sonny did not recognize and could not guess that PWW-10 also illustrated the Pythagorean Theorem even though the diagram used the same notation as PWW-1. As he was examining the diagram he could not right away see the purpose for $(c - a)$.

- I: Can you describe what you see?
- Jerry: A circle, a right triangle ...
- I: Can you guess what this is about?
- Jerry: ... so we forget trigonometry because it is not here. But we can see ... what's this? I'm not sure. What is the usage of $(c - a)$? ... Maybe this is showing some algebraic relations ...

Even when he found out what PWW-10 was about he was still focused on $(c-a)$ and its purpose.

Jerry: Oh... if we want to prove the Pythagorean Theorem ...
[long pause]

I: What are you thinking?

Jerry: I'm thinking about ... I'm thinking about to use the $(c-a)$, but I haven't figured out how $c-a$ can be applied.

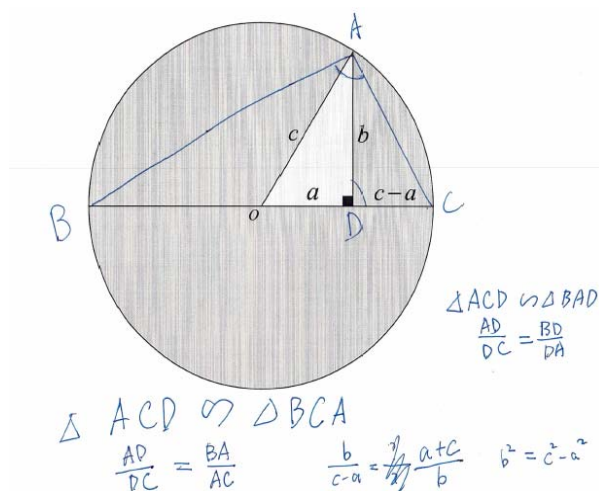


Figure 35 – Jerry's work on PWW-10

When offered a hint to connect points B and C with vertex A , Jerry recognized that he can use similar triangles to obtain the desired results; however, he offered no justification for the similarity of the triangles used. Moreover, Jerry first decided to use $\triangle ACD$ and $\triangle BAD$, which were in “standard position”, to set up ratios of sides then he used $\triangle ACD$ and $\triangle BCA$ to set up ratios of sides, but he justified the similarity of these triangles by transformation.

Jerry: Okay. So, $\triangle ACD$ is similar to ... let's flip it, okay so A corresponds to B ... and then D corresponds to A ... and C is just C

I: But how do you know they are similar?

Jerry: Because I do it in order

To solve PWW-7, Jerry tried to use trigonometric identities to derive desired results. He first found $\sin 2\theta = 2\sin\theta\cos\theta$ using the area of $\triangle ACB$ and then he proceeded to derive $\cos 2\theta$.

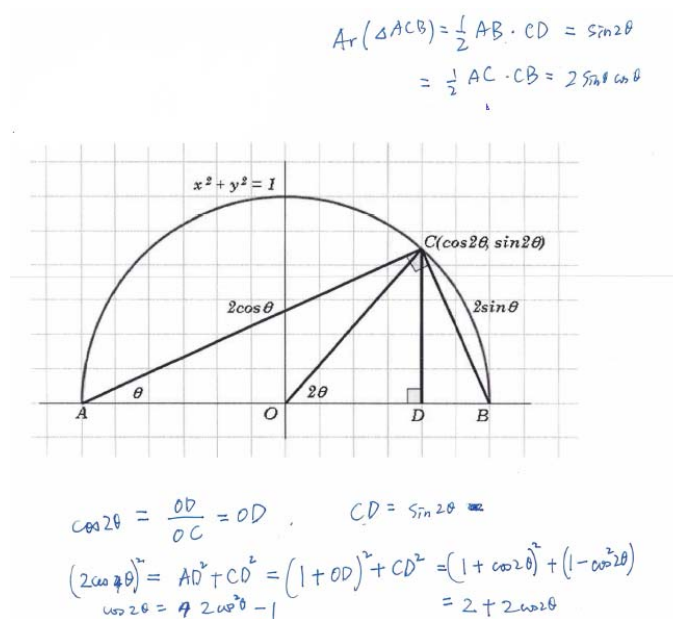


Figure 36 – Jerry's work on PWW-7

Jerry did not observe that certain information was given in the diagram already, i.e., $C(\cos 2\theta, \sin 2\theta)$, and proceeded to derive it. He then used the Pythagorean Theorem to find desired results. Jerry chose to perform algebraic manipulations to obtain the solution rather than use geometric relationships (similarity of triangles) which would require identification of corresponding sides but result in fewer algebraic manipulations.

When solving other visual theorems Jerry was quick in recognizing them, but used numeric expressions to explain these results; although when solving PWW-5 he did present a procedure similar to a proof of geometric series convergence.

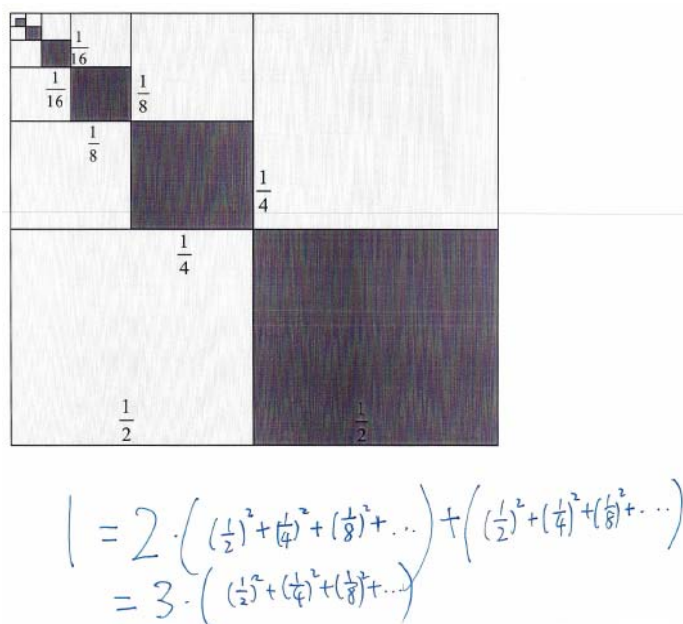


Figure 37 – Jerry’s work on PWW-5

MATTHEW was educated in the United States. His first and only language was English. Upon graduation from the teacher certification program he intended to teach high school mathematics. He held a bachelor's degree in mathematics with no minor concentration and he had limited teaching experience, only that, which Practicum afforded him. In high school Matthew learned geometry with trigonometry but not as a separate course the material was taught as part of the integrated mathematics. Regarding time allocated to various content areas in his high school program Matthew replied: “I just remember algebra, lots of algebra and also graphing linear and quadratic equations”. Matthew did not remember doing “ruler and compass” constructions, when asked about it he replied: "... Do you mean like paper constructions? ... Like origami?". Neither did he remember doing any proofs. However, Matthew did take one semester of geometry in his undergraduate program and two semesters of linear algebra. In his graduate program Matthew took one semester of problem-solving and mathematics methods.

Matthew scored (5, 5, 5, 3, 5) respectively on the van Hiele Geometry Test and therefore, his level attained was assigned vHL3. Based on the test-item analysis Matthew has difficulty differentiating between a proposition and its converse, and he does not distinguish between necessary and sufficient conditions. If van Hiele levels were assigned based on the weaker criterion (WS) Matthew would have attained vHL5.

During clinical interview Matthew was able to recognize most of the visual theorems, but he did not know how to prove them.

Matthew: I do not know how to prove it ... I can do proofs when I know what is given and what I need to find ...

I: This situation is similar except you have to read from the diagram what is given and you already know what you need to prove ...

Matthew: Yeah, but how do I know what is on the diagram?

When given sufficient hints Matthew was able to prove PWW-1. Matthew created a solution sketch, but he did not offer a plan. Even though he was asked to describe what he sees in the diagram he had difficulty describing it. He would list objects, but not their relationships.

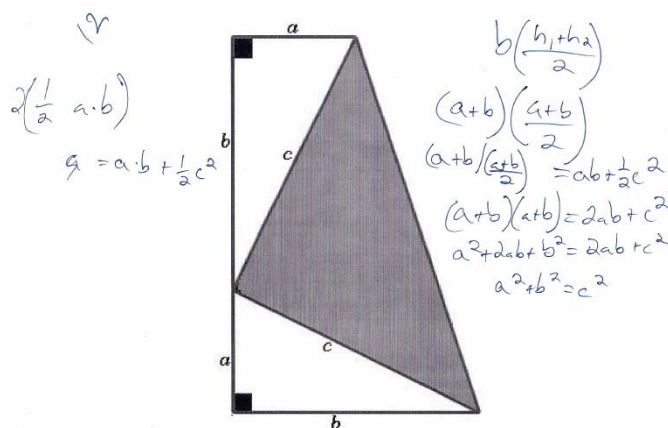


Figure 38 – Matthew's work on PWW-1

Matthew: I see trapezoid, triangles, right angles, letters that stand for measures of sides.

I: Can you now describe these angles, sides, triangles, and the trapezoid?

Matthew: Yeah, two triangles are white and one is blue? Two angles are right angles, but there are other angles in these triangles they are acute.

When solving PWW-10 Matthew realized that it was similar to the problem he has seen a few slides back and was able to do additional construction and choose the method for solving it based on his previous experience; however because he only partially solved PWW-7 he did not know how to proceed with the solution of PWW-10 and eventually abandoned it. He was satisfied with the fact that he know what the problem was about and that to solve it he needs to use similarity of triangles. Moreover, he even gave a justification for the similarity of triangles.

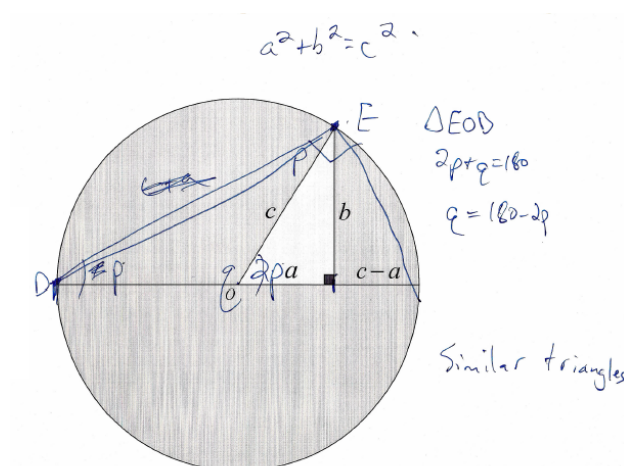


Figure 39 – Matthew's work on PWW-10

When working on PWW-7 Matthew was able to express $\sin 2\theta$, but not the $\cos 2\theta$. He established a similarity between two triangles, but not based on properties of similar

figures, but based on rotation. He drew the triangles under examination in standard position and thus was able to establish the correspondence of sides.

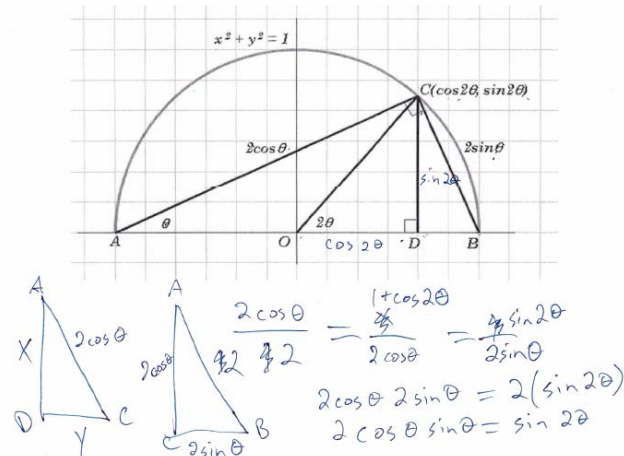


Figure 40 – Matthew's work on PWW-7

Matthew was able to recognize all other visual theorems and write them in symbolic notation; however he did not offer any proofs.

CHAPTER 5 - DISCUSSION

In this section we will consider evidence gathered throughout the study regarding pre-service high school mathematics teachers' academic experience, visual spatial skills, geometry knowledge, diagrammatic reasoning skills, and their geometry knowledge for teaching. First we attempted to determine whether there is a relationship between visual-spatial skills, academic experience, and geometric knowledge as measured by the van Hiele model. Data obtained through questionnaire and testing was processed and correlation coefficients were computed.

Visual-spatial Skills and van Hiele Levels

Since diagrammatic reasoning skills may be modality dependent PSVT was selected for this study because it measures participants' spatial rather than analytic ability to manipulate objects in their "mind's eye". Nine study participants tested above 70%, which is considered moderate to high skill level. If the study participants experienced difficulty in reasoning from the diagrams, for those that scored above 70% on the PSVT, modality was not a contributing factor. Furthermore, data from this study showed that there exists a positive correlation ($r = 0.854$) between visual-spatial skills as measured by the PSVT and participants' attainment of the van Hiele levels. However, because the sample population was small $n = 12$ the confidence in the results is very small. Greater sample is needed to establish with greater degree of confidence if indeed there is a relationship between visual-spatial skills and levels of geometric knowledge of pre-service high school mathematics teachers.

Visual-spatial Skills and Academic Experience

The geometry score was computed based on the reported data. Two possibilities were considered, namely, either pre-service mathematics teachers with stronger visual-spatial skills select an academic program that has greater number of courses in spatial relations, i.e., they will have a natural affinity towards geometry, topology, etc; or prospective teachers with higher PSVT score have achieved better scores because they had more learning opportunities in spatial relations. The correlation coefficient of $r=0.111$ was obtained. Analysis of the data showed no correlation between the geometry score (GS) and the visual-spatial skills measured by the PSVT of the pre-service mathematics teachers. Looking at individual data results the geometry score (GS) ranged between 4 and 9 and the visual-spatial score (PSVT) ranged between 12 and 21. Participants with high geometry score did not consistently achieve higher scores on the PSVT, for example, Nancy with the GS = 8, second highest and PSVT = 17 – a below average score; and Jenny with the GS = 4, the lowest in the group and PSVT = 19 above the group average.

Academic Experience and van Hiele Levels

The geometry scores (GS) reflected participants' academic experience with courses dedicated to spatial relations. They were compared with the participants' corresponding *attained van Hiele* levels. Cited research shows that van Hiele levels of geometric thought are attained through instruction. It then follows that richer academic experience in mathematics in general and in geometry in particular should correlate with higher levels of geometric thinking. According to the result of this study there is no correlation ($r = 0.088$) between academic experience in geometry and attained van Hiele levels among pre-service

high school mathematics teachers. Clearly, there is a contradiction between the finding in this study and the body of accumulated research; after all attainment of van Hiele levels occurs through instruction. However, there is a possible explanation for such results. Firstly, data regarding academic experience was anecdotal and may not be as reliable as data collected on standardized tests. For example, during the initial questioning Jerry reported taking 24 mathematics courses in his undergraduate program. Only after being specifically questioned during his interview regarding this number, did Jerry explain that these were not semester courses, but 24 trimesters of instruction. Even that number seemed inflated when compared with Peter's responses; both were educated in the same system at similar institutions, Jerry's numbers seemed unusually high. Additionally, in the case of Mandy, who reported that she took as many mathematics courses as it was necessary to become a good teacher, information was not possible to obtain. Secondly, in general, study participants had limited academic experience with courses in spatial relations during their undergraduate studies, even those participants who were assigned the highest geometry scores in the group, i.e., Jerry (GS = 9) and Nancy (GS = 8), had limited academic experience in studying spatial relations. Conversely, Timmy had four courses in his academic career dedicated to spatial relations, but his geometry score was (GS = 6), a below-average score. Timmy was lacking in geometry foundation, acquired in middle school and high school. Finally, 58% of participants in this study passed vHL5 even though they did not attain it. This occurred because their results did not satisfy the fixed-sequence property of the model. vHL4 – Deduction and vHL5 – Rigor describe geometric knowledge acquired at the undergraduate level of studies. Since the majority of participants did not take a Geometry course in their undergraduate programs their academic experience

was limited to the knowledge acquired in high school. Since according to the properties of the model, learners should internalize vHL4 tasks to function on vHL5, and since none of the participants acquired vHL4 they have learned the ideas of vHL5 by rote, i.e., the level-reduction phenomenon. (Usiskin, 2011) Therefore, it may not be possible to assess the relationship between academic experience and van Hiele levels based on the coursework participants have done.

Diagrammatic Reasoning and van Hiele Levels

The purpose of this study was to learn whether there exists a relationship between geometric knowledge and visual reasoning skills of pre-service high school mathematics teachers. According to Fischbein visual reasoning organizes information in meaningful structures and it is an important tool in analytical development of a solution. (1987, p.101) According to Arcavi interpretation of visual information is aided by symbolic representation and verbalization is a significant factor (1994). Reading diagrams requires the knowledge of certain representational syntax and semantics. Furthermore, according to our definition diagrammatic reasoning involves observing graphically rendered cognitive constructs, perceiving their components and inherent structures, reflecting upon these perceptions, intuitively generating and verifying new hypotheses and lastly, making connections. In view of this definition let us return to the discussion of the van Hiele model and the vHL's attained by the study participants.

The description of the van Hiele model (Usiskin, 1982; Mayberry, 1983; Senk, 1983; Mason, 1997) indicates that subjects functioning at vHL4 understand the role of definitions, axioms, and theorems in mathematical justifications. They are capable of

differentiating between inductive and deductive arguments; they can differentiate between necessary and sufficient conditions. Moreover, subjects functioning at vHL5 perceive mathematics as a study of models. They recognize and can discuss the merits of using various methods of proof, for example, indirect proof or proof by contradiction. At this level mathematical arguments may be stripped of their content and argued formalistically. A counterintuitive proof will be accepted if the argument is valid. The problems presented to the participants in this study were not sophisticated enough to observe consistently vHL4 and vHL5 behavior. However, they were complex enough to observe vHL1, vHL2, and vHL3 behaviors.

Let us recall that eight out of twelve pre-service teachers participating in this study, when tested, achieved scores corresponding to vHL3 attainment; therefore, according to the model they should have been able to perceive relationships between geometric objects and between the properties of these objects. At this level subjects should be able to create meaningful definitions and they should be able to justify object properties by giving informal arguments. Reviewed literature states that at this level subjects still may not recognize the need to prove certain properties since they may seem to the subjects self-evident, meaningless, or useless. However, of the twelve study participants seven achieved “stricter criterion” score in vHL5 – Rigor, which is characterized by the understanding of the axiomatic nature of mathematics and the need for proof. On the one hand the study participants understand the essence of mathematical inquiry; on the other hand they apply this standard globally, but not locally as in geometric problem solving. There lies a contradiction. According to Professor Usiskin (2011) this group of pre-service teachers

does not fit the van Hiele model, a phenomenon previously described in the literature. (Usiskin, 1982; Mayberry, 1983; Senk, 1989)

Let us consider the solutions to visual theorems of the following participants: Matthew, Ashley, Abigail, and Timmy since they attained vHL3 based on “stricter criterion”, i.e., they passed all levels up to and including vHL3 with the score of at least 4 and they passed vHL5 according to “stricter criterion”. Based on the transcripts of their interviews all four had difficulties reasoning from the diagrams characteristic of vHL1 – vHL3. For example, Matthew could not identify properties of geometric objects based on the information provided on the diagrams. He could list the objects by their physical properties such as “white triangles” and “blue triangle”, but did not identify them as right triangles or an isosceles triangle respectively even though this information was clearly indicated on the diagram. Ashley knew about the Pythagorean Theorem, but associated it with the specific diagram, she called “normal”, which “has boxes” on the sides of the triangle. Identifying a square with a box is a vHL1 characteristic. Timmy consistently called a square with the area a^2 , indicated on the diagram, a rectangle. Abigail called a trapezoid a rectangle, or an isosceles triangle an equilateral triangle. This indicates that they did not function on vHL2 or vHL3 and there was a lack of knowledge of the representational syntax and semantics. None of these four participants felt compelled to give justification for the similarity of triangles – a required characteristic of vHL3; they simply assumed the similarity by inspection.

With respect to visual theorems of algebra all participants, except Mandy (vHL1) and Jenny (GS = 4, vHL3) were able to recognize the “algebra tile”-like quality of the

diagrams. Jenny could not write down the symbolic expressions of these theorems and could not algebraically “prove” them. However, Mandy, once told what the visual theorem was about, was able to write down its symbolic expression and perform algebraic manipulations to “prove” it.

With respect to the presented visual theorems in number theory, all participants, except Jenny and Mandy, were able to interpret the given diagrams and write down numeric interpretation of the picture in front of them. Barbara (vHL1), Mandy (vHL1), Nancy (vHL3), and Matthew (vHL3) were able to produce statements in symbolic form, but did not offer to prove these results, generally proved using the method of mathematical induction.

With respect to the presented visual theorems of geometric sums, all participants, except Jenny and Mandy, were able to interpret the given diagrams and write down numeric expressions which described the pictures in front of them. Timmy attempted to come up with a symbolic expression, but did not complete it. Jerry attempted to produce a proof of one of the problems, but worked with a numeric, not a symbolic expression.

With respect to the presented visual theorems of trigonometric identities, one of the visual theorems was intuitively obvious. To solve it required nothing more than reading the geometric notation presented in the diagram and expressing the tangent of the given angle as a ratio of relevant sides. Abigail, Jenny, and Mandy were unable to interpret the symbolic notation of the diagrams. However, Mandy was able to state and show the trigonometric identities when told what the diagrams were about. Jerry ignored the

diagrams when asked to prove these theorems. He remembered both identities and was able to write down and “prove” both of them. The rest of the group struggled with the similarity of triangles, that is, the participants did not prove similarity and did not use the “corresponding angles” property to establish corresponding sides so that they could find the ratios of sides – a violation of vHL3. The participants attempted to re-draw similar triangles in “standard position” to establish correspondence.

Geometric Knowledge for Teaching

When comparing conceptual maps of experts working on visual theorems and pre-service teachers’ solutions, the issue of schema organization requires attention. According to Dubinsky (1991), individual’s thinking undergoes re-organization as she engages with a given task. Therefore, uncovering individual’s cognitive schema may be done only by observing consistent responses across multiple tasks. In experts’ concept maps two distinct organizations were observed:

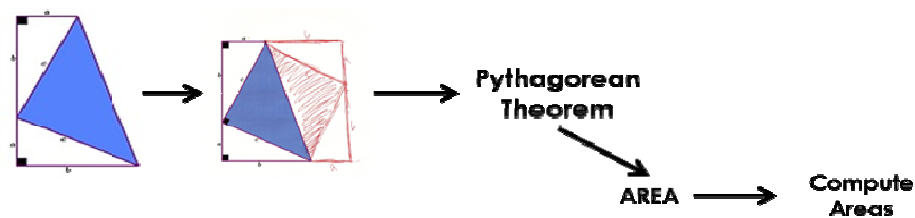


Figure 41 – PWW1 Experts’ Schema Organization (version 1)

Presented diagram triggered activation of memory – recognition of previously established result, organized under the schema of “Pythagorean Theorem” of which Area is a sub-schema. Once the expert recognized what this diagram was “about”, computing areas was an exercise in algebraic manipulation. The important first step was to observe the diagram and search memory for a related construct.

The second method of finding a solution for the presented visual theorem is illustrated by this diagram:

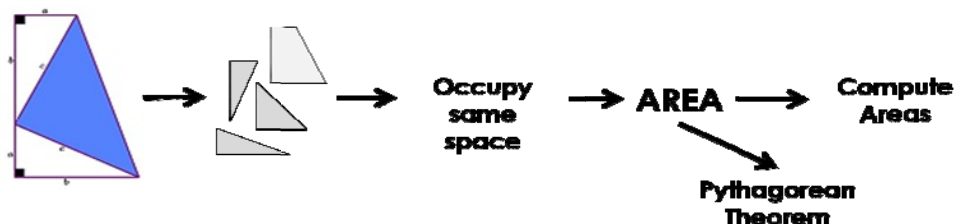


Figure 42 – PWW1 Experts' Schema Organization (version 2)

In this case the experts decomposed the diagram into parts and identified a physical relationship of “occupy same space” or “tiling” between these parts. This, in turn, activated the “Area” schema of which “Pythagorean Theorem” was a sub-schema. Once the relationship was established through algebraic manipulation Pythagorean Theorem was proven. In contrast the following diagram illustrates pre-service teachers approach to proving this visual theorem:

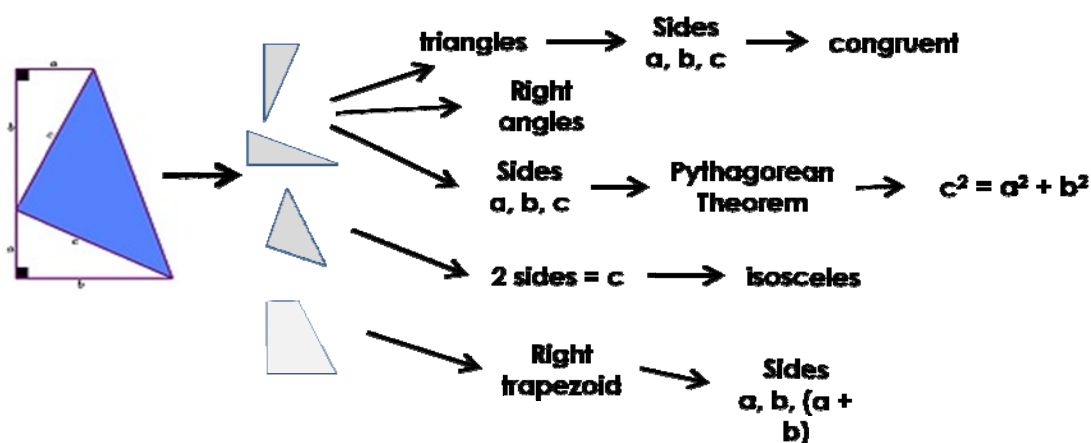


Figure 43 – PWW1 Pre-service Teachers' Schema Organization

The majority of pre-service teachers parsed the original diagram into triangles and trapezoid components from the start, without trying to understand the intent of the visual

theorem. When asked “Can you say what you see?” the responses were “right triangle”, “shaded triangle”, “right angles”, and so on. As a result they missed the unifying schema and focused on the properties of objects rather than the properties of relations. Moreover, since the participants’ geometric knowledge described by the van Hiele model was at the lower levels of the model their parsing of the diagrams into lower sub-schemas was consistent with their levels of geometric thought.

CHAPTER 6 – SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

What should teachers know to be better Mathematics teachers? This study was inspired by this researcher's experience as a teacher-educator seeking understanding of factors relevant to the improvement and enrichment of teachers' knowledge for teaching.

Summary

Diagrammatic reasoning is a newly re-discovered area of research in Mathematics Education possibly due to the changing nature of mathematical practice and mathematics learning. Although the formal nature of mathematical practice remains unchanged, the development of new tools for doing and learning mathematics require skills previously not considered essential, i.e., visual or diagrammatic reasoning. There is no subject matter in secondary or post-secondary education that systematically develops diagrammatic reasoning in students, except Geometry. While NCATE/NCTM program standards call for teachers to use diagrammatic reasoning in teaching mathematics and suggest dynamic geometry environments as tools for teaching geometry, not all prospective mathematics teachers are adequately prepared for the task. This study attempted to explore a possible relationship between geometric knowledge and diagrammatic reasoning of pre-service high school mathematics teachers. More specifically, this study attempted to gain understanding of the relationship between van Hiele levels of geometric thought and pre-service mathematics teachers' ability to prove/explain visual theorems.

The study sought answers to the following research questions:

1. Is there a relationship between visual-spatial skills, van Hiele levels of geometric thought, and academic experience in Geometry among pre-service secondary mathematics teachers?
2. Is there a relationship between van Hiele levels and diagrammatic reasoning skills of pre-service high school mathematics teachers as they prove/explain visual theorems?
3. Does van Hiele model adequately predict pre-service high school mathematics teachers' knowledge of geometry?
4. Does van Hiele model adequately predict pre-service high school mathematics teachers' knowledge of geometry for teaching?

In the course of the study twelve volunteers, pre-service mathematics teachers, were surveyed regarding their academic experience, tested for their visual-spatial skills, tested on their geometric knowledge, and given graphically rendered theorems from secondary mathematics curriculum to reason from and prove in the sense of generating explanations. The results were recorded, processed, and analyzed. The conclusions are as follows:

Conclusions

With respect to Research Question 1: Is there a relationship between visual-spatial skills, van Hiele levels of geometric thought, and academic experience in Geometry among pre-service secondary mathematics teachers?

This study found no relationship between visual-spatial skills and academic experience with respect to the study of geometry among pre-service mathematics teachers. Participants who scored in the high range of the Purdue Spatial Visualization Test were just as likely to have taken courses and studied spatial relations as those who have scored in the low range.

This study found no relationship between pre-service teachers' academic experience and attained van Hiele levels of geometric thought, contrary to the property of the van Hiele model, which states that progression through the levels occurs by means of instruction. It was expected that participants with higher geometry scores would attain higher van Hiele levels; however, the obtained result may be explained by several factors, among them: anecdotal self-reporting of academic experience by the participants; insufficiently high geometry score, or the lack of geometry experience at the undergraduate level where participants would gain experience in doing geometry at the level of deduction (vHL4) and rigor (vHL5).

This study found a correlation between visual-spatial skills and the van Hiele levels attained by the participants; however, since there were only 12 participants in the study the results may not be reliable enough. There is a possibility that the results were not varied enough to warrant a conclusion of the relationship.

With respect to Research Question 2: Is there a relationship between van Hiele levels and diagrammatic reasoning skills of pre-service high school mathematics teachers as they prove/explain visual theorems?

Through observation, analysis, and comparison of interviews this study found a correspondence between the diagrammatic reasoning of pre-service mathematics teachers and behaviors attributed to certain van Hiele levels. For example, naming a square “a box”, naming a square “a rectangle”, identifying triangles by their color in the diagram rather than by their geometric properties – behaviors typical of the vHL1 – Visualization, whereby recognizing geometric objects is done by appearance only. Other behaviors exhibited by the pre-service teachers such as not recognizing the need for justification of certain geometric properties, such as similarity or congruence, attest to the pre-service teachers geometric knowledge at vHL2 – Pre-deduction. Moreover, the study participants that function at a higher van Hiele level were more skilled at recognizing visual theorems and “proving” them. Based on this analysis it is possible to conclude that there exists a relationship between geometric knowledge and visual reasoning skills among pre-service mathematics teachers that participated in this study.

With respect to Research Question 3: Does van Hiele model adequately predict pre-service high school mathematics teachers’ knowledge of geometry?

In her (1989) study Senk argued that vHL3 is a good predictor of students’ proof-writing skills. Achieving vHL3 meant students will be successful proof-writers and knowledge below vHL3 meant lack of success in proof-writing. To that extent this study confirms Dr. Senk’s result. In this study in general, pre-service teachers exhibited behaviors that would characterize them as poor proof-writers. For example, they lacked knowledge of geometric syntax and semantics, they had difficulties distinguishing between a proposition and its converse, they constructed circular arguments proving what they have assumed at the start,

accepting a specific computation as a general proof, etc.; yet, the majority attained vHL3 and even scored maximum points at the higher levels. One possible explanation is that they were not properly assigned attained van Hiele Levels. However, this researcher followed Professor Usiskin's method (1983) of determining level attainment, therefore, it is possible that this van Hiele model may adequately tests students in the process of instruction, but when it comes to assessing teachers' knowledge of geometry this instrument by itself is inadequate because it does not take into consideration diagrammatic reasoning skills.

With respect to Research Question 4: Does van Hiele model adequately predict pre-service high school mathematics teachers' knowledge of geometry for teaching?

In their (2000) study Chinnappan and Lawson argued that successful geometry problem solvers are characterized by their spontaneous abilities to access geometric rules. (p. 39) Schoenfeld (1988), Robinson, Even & Tirosh (1992), Chinnappan & Lawson (2005) argued that the quality of teachers' knowledge for teaching is characterized by the richness of the interconnected schemas of their subject-matter content knowledge.

This study has demonstrated that pre-service mathematics teachers who participated in this study did not possess rich representational schemas needed for diagrammatic reasoning. The participants performed well on algebraic algorithmic tasks if they were presented in a familiar context. They had difficulties translating between diagrammatic and symbolic representations, they did not "skip-stepped" when parsing diagrams, they did not identify problem spaces, and they had difficulties making connections. However, none of these characteristics of their geometric knowledge for teaching would come to light if they were tested using the van Hiele model alone or even if they were tested by

means of a geometric knowledge test. As was argued in the preceding pages, geometric knowledge may be acquired by rote.

Recommendations

Attempting to gain understanding of a relationship between pre-service mathematics teachers' knowledge of geometry and their diagrammatic reasoning skills several factors must be taken into consideration. When designing a study one must be cognizant of the role of the interviewer conducting an inquiry and the effect one has on the interviewees. A more structured script would provide a greater control of the experiment. A video-recording would also offer greater insight into the body language of interviewees. A bigger sample size would allow statistical analysis, and therefore, greater insight into the results, not possible with a group of twelve participants. In retrospect, a choice of fewer visual theorems with deeper analysis of each would produce results. In order to obtain more definitive results with respect to the correlational study a deeper, more detailed questioning regarding academic experience might have helped. Finally, the scope of this study did not permit considering the role of language in diagrammatic reasoning, which might have been a factor in pre-service teachers' performance. This warrants a future study.

This study underscores the benefits of using diagrammatic reasoning as an assessment tool. It can help teachers gain insight into students' understanding of mathematical concepts, the degree of concept acquisition, and the ability to communicate about mathematics. In case of Geometry, diagrammatic reasoning can augment teachers' understanding of students' van Hiele levels of geometric thought. Such knowledge will inform and improve their practice.

Moreover, this study highlights certain deficiencies in teacher education and the need for re-examination of academic requirements and curricular changes needed to remedy these deficiencies. Teacher education programs must consider not only ways to improve prospective teachers' knowledge of geometry, but also ways to increase teachers' knowledge of geometry for teaching. Since there is a renewed emphasis on teaching Geometry across the divisions a methods course dedicated to this subject might improve pedagogical content knowledge in geometry.

The research literature to a great degree and this study to a lesser degree show that since there is a renewed interest in diagrammatic reasoning, there is a need for an assessment instrument, which will help us develop greater understanding of pre-service and in-service mathematics teachers' geometric knowledge and their diagrammatic reasoning skills. Such instrument will be useful specifically because of the NCTM mandate to use dynamic geometry environments in teaching and learning of Geometry, yet we know little of the students' or their teachers' diagrammatic reasoning skills when learning or teaching geometry with technology.

Moreover, as Freudenthal (1973) and Banchoff & Wermer (1992) suggest and what is evident in this study is that the deficit in the foundation of the study of Geometry is not compensated by higher-level, well-organized, globally-axiomatic courses in spatial relations. Pre-service mathematics teachers experience many more courses in algebra and higher mathematics than in Geometry yet they are expected to teach Geometry and they are expected to develop pedagogical content knowledge without any Geometry methods courses. This warrants further study.

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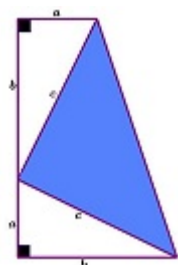
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APPENDICES

Appendix – Composite Expert Solutions to PWW's

PWW-1
Pythagorean
Theorem



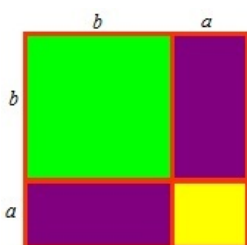
- The problem is about Area.
The sum of areas of the three triangles should equal the area of the trapezoid. Measures of the sides are given for two right triangles; need to prove that the third triangle is a right triangle. It is by complementary angles. Therefore,

$$2 \times \frac{ab}{2} + \frac{c^2}{2} = \frac{a+b}{2}(a+b)$$

- The diagram is a half of the known proof of the Pythagorean Theorem

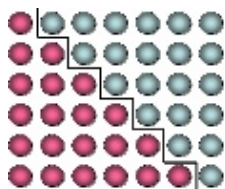
$$4 \times \frac{ab}{2} + c^2 = (a+b)^2$$

PWW-2
Square of a
sum



The area of a square with side $(a + b)$ consists of the area of a square with side b , the area of the square with side a and the area ab of two remaining rectangles.

PWW-3
The sum of
 n integers

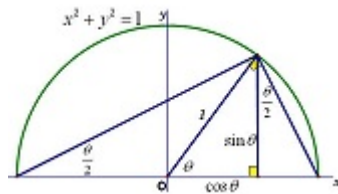


The zigzagged line cuts the rectangle in halves, i.e., the same number of dots in the bottom-left portion as in the upper-right portion of the rectangular formation. There are $1 + 2 + 3 + 4 + 5 + 6 = 21$ pink dots and the same number of the blue dots and together there are 42 dots in this rectangular array. On the other hand the number of dots in this array may be computed by multiplying number of rows by the number of columns in the array. Then

$$2(1 + 2 + 3 + \dots + n) = n(n+1)$$

$$\therefore 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

PWW-4
The half-angle
tangent
formula



- Tangent of an angle expressed as ratio of given sides. Given a right triangle, tangent of an acute angle is a ratio of the side opposite to it and the side adjacent to it.

From the diagram $\tan \frac{\theta}{2}$ may be expressed in two ways (using different triangles).

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} \text{ or } \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$$

- Know the formula
- The sum of length of sides of the green squares is 1. From the diagram the sides of the green squares are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$.

The common ratio is then $\frac{1}{2}$. By

performing the following computations we obtain the geometric sum:

$$s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$\frac{1}{2}s = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$s - \frac{1}{2}s = \frac{1}{2} \Rightarrow 2s - s = 1 \quad \therefore s = 1$$

- The sum of areas of the green squares is $\frac{1}{3}$. From the diagram the areas of the

green squares are $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$. The

common ratio is then $\frac{1}{4}$. By performing

the following computations we obtain the geometric sum:

$$s = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$$

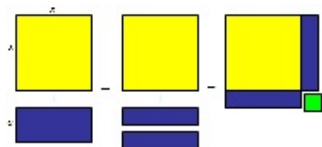
$$\frac{1}{4}s = \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots$$

$$s - \frac{1}{4}s = \frac{1}{4} \Rightarrow 4s - s = 1 \quad \therefore s = \frac{1}{3}$$

PWW-5
Geometric
Sum



PWW-6
Completing
the square



- The green square is “subtracted” from the square formed by adding the original square and the rectangle. From the diagram we have a square with side x and a rectangle with sides x and a . We cut the rectangle into two congruent

rectangles with sides x and $\frac{a}{2}$. The plus

signs imply both addition and grouping. We group the square with the new rectangles in such a way that a new

square is formed with side $\left(x + \frac{a}{2}\right)$ and

its area is $\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$ which is

equal to the area of the original square and rectangle $x^2 + ax$

- Given a semi-circle centered at the origin O with radius 1. A right triangle inscribed in the semi-circle so that its hypotenuse AB is the diameter of the semicircle and its vertex lies on the circumference at $C(\cos 2\theta, \sin 2\theta)$. An altitude is dropped from

$C(\cos 2\theta, \sin 2\theta)$ onto the diameter at D . Radius of the semi-circle OC forms an acute angle $\angle COD = 2\theta$ with the x -axis. We need to express $\sin 2\theta$ and $\cos 2\theta$ in terms of given information.

$\triangle ABC$ and $\triangle ACD$ are similar because their angles are congruent. Therefore,

$$\frac{CD}{AC} = \frac{CB}{AB} \quad \text{and}$$

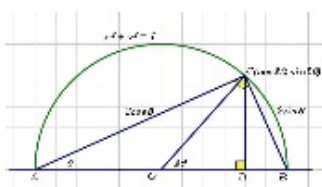
$$\frac{\sin 2\theta}{2 \cos \theta} = \frac{2 \sin \theta}{2} \therefore \sin 2\theta = 2 \sin \theta \cos \theta$$

$$\text{Also, } \frac{AD}{AC} = \frac{AC}{AB} \quad \text{and}$$

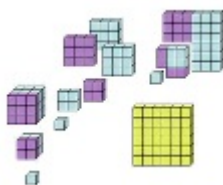
$$\frac{1 + \cos 2\theta}{2 \cos \theta} = \frac{2 \cos \theta}{2}$$

$$\therefore \cos 2\theta = 2 \cos^2 \theta - 1.$$

PWW-7
The double-
angle
formula



PWW-8
The sum of
cubes



- One cube represents a unit. We are given $1^3, 2^3, 3^3$ units. These are broken up and reassembled into a square which contains $36 = 6^2 = (1 + 2 + 3)^2$ units.

$$1^3 + 2^3 + 3^3 + \cdots + n^2 = (1 + 2 + 3 + \cdots + n)^2$$

or $\sum_{i=1}^n i^3 = \left(\sum_{i=1}^n i \right)^2$ We show this result by

using mathematical induction. Given

that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ we want to show that

$$\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2 = \frac{n^2(n+1)^2}{4}. \text{ For}$$

$n = 1$ the case is trivial. We assume it works for n and show that it works for

$$n+1. S_{n+1} = \sum_{i=1}^{n+1} i^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3.$$

After simplifying we get

$$\frac{(n+1)^2(n+2)^2}{4} \text{ which is of the form}$$

$$\frac{n^2(n+1)^2}{4} \text{ for the } (n+1)\text{-th term.}$$

- The sum of the areas of the rectangles adds up to 1. From the diagram, the unit square with area 1 is divided into two rectangles of equal area of $\frac{1}{2}$. One of the rectangles is divided into two squares of equal areas of $\frac{1}{4}$. The

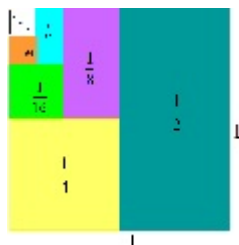
process of subdividing continues so that the area of the unit square may be expressed as

$$A = \left(1 \times \frac{1}{2} \right) + \left(\frac{1}{2} \times \frac{1}{2} \right) + \left(\frac{1}{2} \times \frac{1}{4} \right) + \cdots$$

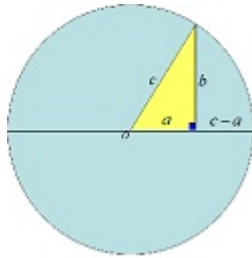
$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1. \text{ Using similar}$$

method for obtaining geometric sum from the previous example, geometric series converges to 1.

PWW-9 the
geometric
sum



PWW-10
Pythagorean
Theorem



- ⊢ Given a circle of radius c with center O .
- ⊢ Is it similar to the double-angle formula?
Additional construction is required.
Connect the ends of the diameter and the vertex of the triangle located on the circumference.
- ⊢ Similar to PWW4 and PWW7
- ⊢

Appendix – Clinical Interview Script

1. Does the participant recognize a given PWW?

- Action:** The participant is shown a PWW without its name.
- Rationale:** If the participant recognizes a given PWW then either it was encountered before, in which case the participant may remember all of it or some of it; may be able to explain the diagram or prove the theorem; or the participant has a certain intuition about it.
- Objective:** To test: curricular knowledge, van Hiele Level 4, geometric intuition.
- Possible outcomes**
- a) Recognizes
 - b) Does not recognize, but guesses correctly
 - c) Does not recognize and guesses incorrectly
 - d) Does not recognize and does not guess

2. Given the name of the “theorem” depicted in the diagram does the participant “see” the theorem in the diagram?

- Action:** The PWW is identified.
- Rationale:** Knowing what the given PWW is “about” may invoke a concept image of the given theorem and may help the participant to begin “thinking aloud” about it.
- Objective:** To invoke: a concept image, procept.
- Possible outcomes**
- a) “Sees” the theorem in the diagram, attempts to explain.
 - b) “Sees” the theorem in the diagram, attempts to explain unsuccessfully.
 - c) Does not “see”, but attempts to make sense of the diagram.
 - d) Does not “see” and does not attempt to make sense of the diagram.

3. Can the participant state the identified theorem?

- Action:** The participant is asked to state the identified theorem.
- Rationale:** To be able to reason from the diagram to recognize geometric objects and their properties one must know what to work towards. Additionally, provides the investigator with an opportunity to offer a “scaffolding remark”
- Objective:** To test: curricular knowledge, van Hiele levels, procept

- Possible outcomes**
- a) Correctly states the theorem; writes a symbolic statement.
 - b) Correctly states the theorem; does not write symbolic statement.
 - c) States the theorem incorrectly; writes symbolic statement.
 - d) States the theorem incorrectly; writes symbolic statement incorrectly.
 - e) Does not know the theorem.

4. Can the participant describe the diagram and through this description develop leads towards a solution?

Action: The participant is asked to describe the diagram.

Rationale: If the participant perceives geometric objects as collections of properties then while describing the image these properties may be invoked and connections made to explain a given PWW.

Objective: To test van Hiele levels.

- Possible outcomes**
- a) Describes the diagram without help, identifies properties
 - b) Describes the diagram without help, does not identify properties
 - c) Describes the diagram with help, identifies properties
 - d) Describes the diagram with help does not identify properties
 - e) Does not describe the diagram.

5. Given a solution in symbolic notation can the participant connect it to the diagram?

Action: The participant is shown a solution of the given PWW in symbolic form and asked to relate it to the diagram.

Rationale: If the participant has difficulties reasoning from the diagram a symbolic statement may be participant's concept image of the given PWW.

Objective: To test: concept image, procept, and multiple representations.

- Possible outcomes**
- a) Demonstrates understanding of proof written in symbolic notation and relates it to the diagram.
 - b) Demonstrates understanding of proof written in symbolic notation but needs help relating it to the diagram.
 - c) Needs help in decoding the symbolic notation but can relate it to the diagram.
 - d) Needs help in decoding the symbolic notation and needs help relating it to the diagram.

6. Once the diagram has been decoded and proven does the participant offer an alternative solution?

Action: The participant is asked to offer an alternative solution.

Rationale: Will offer the participant an opportunity to express his or her attitude towards diagrammatic reasoning.

Objective: To test: making connections and multiple representations

- Possible outcomes**
- a) Offers an alternative solution, explains, and writes it down using symbolic notation.
 - b) Attempts an alternative solution, explains it, but fails to writes it in symbolic form.
 - c) Attempts an alternative solution, writes it in symbolic form, but fails to explain it
 - d) Attempts an alternative solution, does not writes it in symbolic form, fails to explain it
 - e) Does not attempt an alternative solution

Appendix – van Hiele Geometry Test Scores

	Level 1						Level 2						Level 3						Level 4						Level 5					
	1	2	3	4	5		6	7	8	9	10		11	12	13	14	15		16	17	18	19	20		21	22	23	24	25	
Matthew	1	1	1	1	1	5	1	1	1	1	1	5	1	1	1	1	1	5	1	1	0	0	1	3	1	1	1	1	1	5
Jerry	1	1	1	1	1	5	1	1	1	1	1	5	1	1	1	1	1	5	1	1	0	1	0	3	1	1	1	1	1	5
Nancy	1	1	1	1	1	5	1	1	1	1	1	5	1	1	1	1	1	5	1	1	0	0	1	3	1	1	1	1	1	5
Ashley	1	1	1	1	1	5	1	1	1	1	1	5	1	1	1	1	1	5	0	1	0	0	0	1	1	0	1	1	1	4
Jenny	1	1	1	1	1	5	1	1	1	1	1	5	1	1	1	1	1	5	0	1	0	0	0	1	1	0	1	0	1	3
Abigail	1	1	1	1	1	5	1	1	1	1	1	5	1	1	1	1	0	4	0	1	0	0	0	1	1	1	1	1	1	5
Katie	1	1	1	1	1	5	1	1	1	1	0	4	1	1	1	1	0	4	1	1	0	0	1	3	0	1	1	1	1	4
Timmy	1	1	1	1	1	5	1	1	1	1	0	4	1	1	1	0	1	4	0	0	0	0	1	1	1	1	1	0	1	4
Barbara	1	1	1	1	1	5	1	1	0	1	0	3	1	1	1	1	1	5	0	0	1	0	1	2	0	0	1	0	0	1
Peter	1	1	1	1	1	5	1	0	0	1	1	3	1	0	1	1	0	3	0	0	1	0	0	1	0	1	0	1	0	2
Mandy	1	1	1	1	1	5	1	1	0	1	0	3	0	0	0	0	1	1	1	0	1	0	0	2	0	1	1	0	0	2
Polly	1	1	1	1	1	5	0	1	0	1	0	2	1	1	0	0	1	3	1	0	0	0	0	1	0	0	0	1	1	2
Average Group Score			5			4						4.2						1.4						4.0						
Average weighted Group Score			1			2						4						0						16						